Digraph colouring

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Supervisor: Frédéric Havet

Co-Advisor: Stéphane Bessy

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UNIVERSITÉ CÔTE D'AZUR | ÉCOLE UNIVERSITAIRE DE RECHERCHE SYSTÈMES NUMÉRIQUES POUR L'HUMAIN

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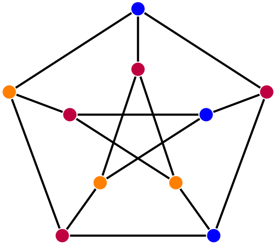
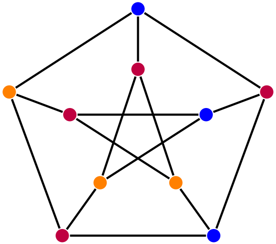
Graph colouring

- **Proper k-colouring** of G: partition of $V(G)$ into k independent sets.



Graph colouring

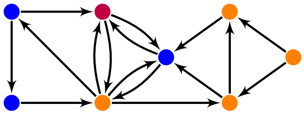
- **Proper k-colouring** of G: partition of $V(G)$ into k independent sets.
- **Chromatic Number** $\chi(G)$: minimum k s.t. G admits a proper k-colouring.



$\chi(G) = 3$

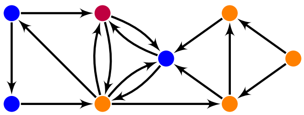
Digraph dicolouring

- ***k*-dicolouring** of D: partition of $V(D)$ into k acyclic subdigraphs (*i.e.* no monochromatic directed cycle).



Digraph dicolouring

- ***k*-dicolouring** of D: partition of $V(D)$ into k acyclic subdigraphs (*i.e.* no monochromatic directed cycle).
- **Dichromatic number** $\vec{\chi}(D)$: minimum k s.t. D admits a k-dicolouring.



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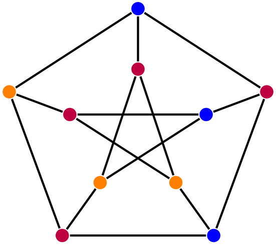

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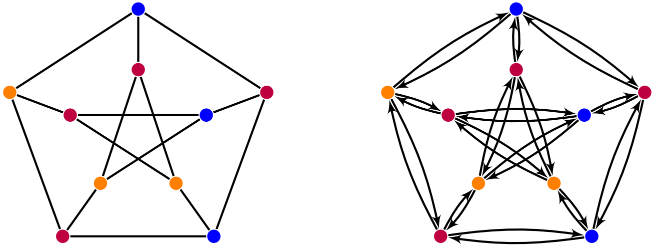
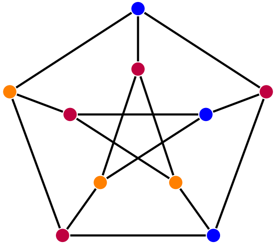


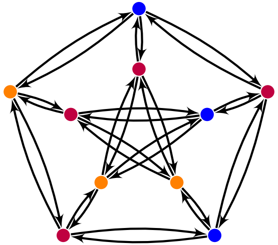
$$\chi(G) = \vec{\chi}(\overleftrightarrow{G})$$

From graphs to digraphs: main questions

Given any result on graph colouring, two questions arise:

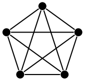
- **Question 1:** Does it generalize to all digraphs?
- **Question 2:** If it does, can we strengthen it on oriented graphs?





From graphs to digraphs: examples (1/2)

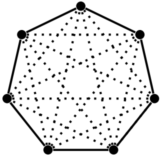
$\omega(G)$: size of a largest clique.
Hole: induced cycle of length at least 4.
Antihole: complementary of a hole.
G is perfect if $\chi(H) = \omega(H)$ holds for every induced subgraph H of G.



Clique

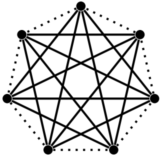
Theorem (Strong Perfect Graph Theorem, Chudnovsky et al. 2006)

A graph G is ***perfect*** *iff G contains neither any* ***odd hole*** *nor any* ***odd antihole****.*



Odd hole

$\omega = 2, \chi = 3$

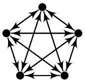


Odd antihole

$\omega = \frac{n-1}{2}, \chi = \frac{n+1}{2}$

From graphs to digraphs: examples (1/2)

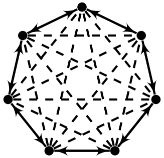
$\overleftrightarrow{\omega}(D)$: size of the largest biclique.
Bidirected hole: a hole in the symmetric part of D.
Bidirected antihole: an antihole in the symmetric part of D.
D is perfect if $\vec{\chi}(H) = \overleftrightarrow{\omega}(H)$ holds for every induced subdigraph H of D.



Biclique

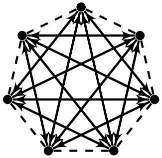
Theorem (Andres and Hochstättler 2015)

A digraph D is ***perfect*** *iff D contains neither any* ***bidirected odd hole****, nor any* ***bidirected odd antihole****, nor any* ***induced directed cycle*** *of length at least 3.*



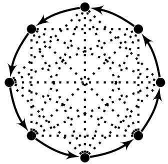
Bidirected odd hole

$\overleftrightarrow{\omega} = 2$, $\vec{\chi} \geq 3$



Bidirected odd antihole

$\overleftrightarrow{\omega} = \frac{n-1}{2}$, $\vec{\chi} = \frac{n+1}{2}$



Induced directed cycle

$\overleftrightarrow{\omega} = 1$, $\vec{\chi} = 2$

From graphs to digraphs: examples (2/2)

A (di)graph is planar if it can be drawn on the plane without crossing edges.

Theorem (Four Colour Theorem, Appel and Haken 1976)

Every planar graph G satisfies $\chi(G) \leq 4$.

Corollary

Every planar digraph D satisfies $\vec{\chi}(D) \leq 4$.

Remark: Best possible because of .

Conjecture (Neumann-Lara 1982)

Every oriented (no digon) planar graph $\vec{G}$ satisfies $\vec{\chi}(\vec{G}) \leq 2$.

Remark: $\vec{\chi}(\vec{G}) \leq 3$ follows from the density of planar graphs.

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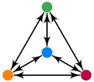
Remark: Best possible because of .

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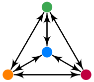
Remark: Best possible because of .

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Contributions

Chordal graphs

- *Dichromatic number of chordal graphs,* [Bessy, Havet, and P., 2023]

On the Directed Brooks' Theorem

- *Brooks-type colourings of digraphs in linear time,* [Gonçalves, P., and Reinald, 2024]
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Minimum density of dicritical digraphs

Definitions

- A graph G is **k-critical** if $\chi(G) = k$ and $\chi(H) < k$ holds for every $H \subsetneq G$.
- A digraph D is **k-dicritical** if $\vec{\chi}(D) = k$ and $\vec{\chi}(H) < k$ holds for every $H \subsetneq D$.

- $g_k(n)$: minimum number of edges in an n-vertex k-critical graph.
- $d_k(n)$: minimum number of arcs in an n-vertex k-dicritical digraph.
- $o_k(n)$: minimum number of arcs in an n-vertex k-dicritical oriented graph.

Trivial bounds

- $d_k(n) \leq 2 \cdot g_k(n)$
- $o_k(n) \geq d_k(n)$

Conjectures (Kostochka and Stiebitz 2020)

1. $d_k(n) = 2 \cdot g_k(n)$ and equality holds only for bidirected graphs (unless $n = k + 1$).
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Minimum density of dicritical digraphs – known results

- **Undirected case:** $g_k(n) \geq \frac{1}{2}(k - \frac{2}{k-1})n - \frac{k(k-3)}{2(k-1)}$. [Kostochka and Yancey 2014]
- **Conjecture:** $d_k(n) \geq (k - \frac{2}{k-1})n - \frac{k(k-3)}{(k-1)}$. [Kostochka and Stiebitz 2020]
  known for $k \in \{2,3,4\}$, open for $k \geq 5$.
- **Best bound:** $d_k(n) \geq (k - \frac{1}{2} + \frac{2}{k-1})n - \frac{k(k-3)}{(k-1)}$. [Aboulker and Vermande 2022]
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Minimum density of dicritical digraphs – contributions

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For every $n, k, p \in \mathbb{N}$ with $n = k + p$ and $2 \leq p \leq k - 1$, $d_k(n) = n(n-1) - 2(p^2+1)$ and equality holds only for bidirected Dirac's graphs.

Theorem (Havet, P. and Rambaud 2023)

For every $n \in \mathbb{N}$, $d_4(n) \geq \frac{10}{3}n - \frac{4}{3}$ and equality holds only for bidirected Ore's graphs.

Theorem (Havet, P. and Rambaud 2023)

For every $n \in \mathbb{N}$, $o_4(n) \geq \left(\frac{10}{3} + \frac{1}{51}\right) n - 1$.

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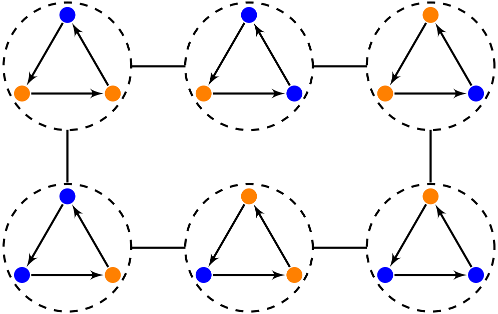
Digraph redicolouring

$\mathcal{D}_k(D)$: the ***k*-dicolouring graph** of D:

- $V(\mathcal{D}_k(D))$ are the k-dicolourings of D,
- $\gamma_i\gamma_j \in E(\mathcal{D}_k(D))$ if $\gamma_i = \gamma_j$ except on one vertex.



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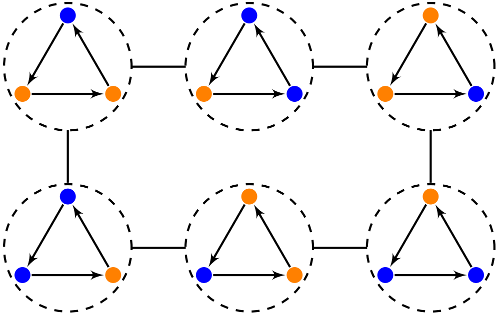

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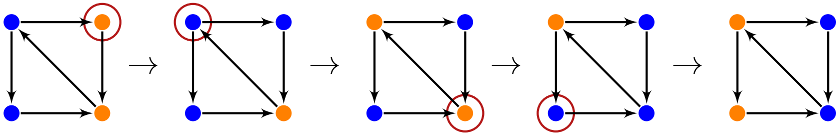
Recolouring sequence: a path in $\mathcal{D}_k(D)$.



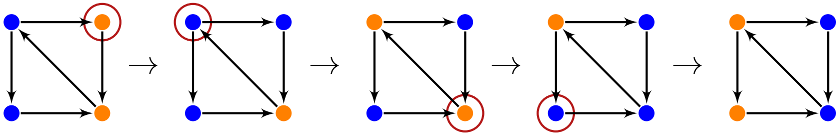
D is ***k*-mixing**: $\mathcal{D}_k(D)$ is connected.

Main questions:

- Is D k-mixing ?
- Can we bound the diameter of $\mathcal{D}_k(D)$?

Digraph redicolouring

Recolouring sequence: a path in $\mathcal{D}_k(D)$.



D is **k-mixing**: $\mathcal{D}_k(D)$ is connected.

Main questions:

- Is D k-mixing ?
- Can we bound the diameter of $\mathcal{D}_k(D)$?

Digraph recolouring – bounded degeneracy

Undirected graphs

$$\delta = \max_{H \subseteq G} \min_{v \in V(H)} d_H(v)$$

- If $k \geq \delta + 2$, then G is k-mixing.

  (Bonsma and Cereceda 2007 ; Dyer et al. 2006)

- Conjecture: $diam(\mathcal{C}_k(G)) = O(n^2)$

  (Cereceda 2007)

- If $k \geq \delta + 2$, $diam(\mathcal{C}_k(G)) = O_\delta(n^{\delta+1})$.
- If $k \geq \frac{3}{2}(\delta + 1)$, $diam(\mathcal{C}_k(G)) = O(n^2)$.

  (Bousquet and Heinrich 2019)

- If $k \geq 2(\delta + 1)$, $diam(\mathcal{C}_k(G)) \leq \delta \cdot n$.

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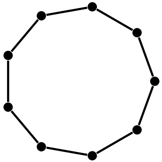
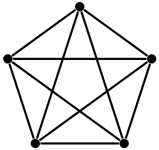
Directed Brooks' Theorem

Theorem (Brooks 1941)

Let G be a connected graph, then $\chi(G) \leq \Delta(G)$ unless G is an **odd cycle** *or a* **complete graph**.



$\Delta = 2, \chi = 3$



$\Delta = n-1, \chi = n$

Directed Brooks' Theorem

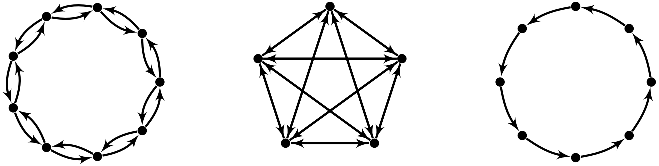
Theorem (Brooks 1941)

Let G be a connected graph, then $\chi(G) \leq \Delta(G)$ unless G is an **odd cycle** *or a* **complete graph**.

$$\Delta_{\max}(D) = \max_{v \in V(D)} \max(d^-(v), d^+(v)).$$

Theorem (Mohar 2010)

Let D be a connected digraph, then $\vec{\chi}(D) \leq \Delta_{\max}(D)$ unless D is a **bidirected odd cycle**, *a* **bidirected complete graph**, *or a* **directed cycle**.



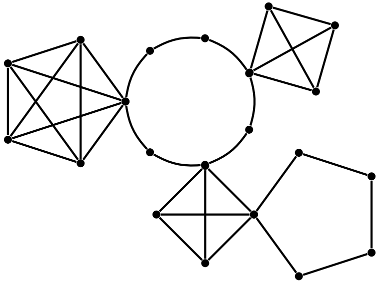
$\Delta_{\max} = 2,\ \vec{\chi} = 3$ $\Delta_{\max} = n-1,\ \vec{\chi} = n$ $\Delta_{\max} = 1,\ \vec{\chi} = 2$

Generalisations of Brooks' Theorem (1/2)

List assignment L: a list of available colours $L(v)$ for every vertex v.
L-colouring: a colouring α s.t. $\alpha(v) \in L(v)$ for every vertex v.

Theorem (Borodin ; Erdős et al. 1979)

Let G be a connected graph and L a list assignment such that $\forall v \in V(G)$, $|L(v)| \geq d(v)$. If G is not L-colourable, it must be a **Gallai tree**.



Generalisations of Brooks' Theorem (1/2)

List assignment L: a list of available colours $L(v)$ for every vertex v.
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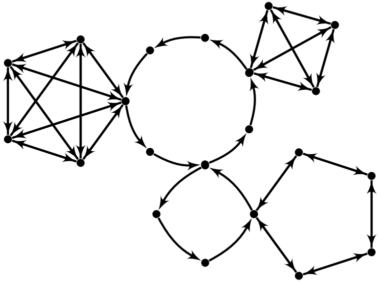
Theorem (Harutyunyan and Mohar 2011)

Let D be a connected digraph and L a list assignment such that $\forall v \in V(D)$, $|L(v)| \geq \max(d^-(v), d^+(v))$. If D is not L-dicolourable, then it is a **directed Gallai tree**.



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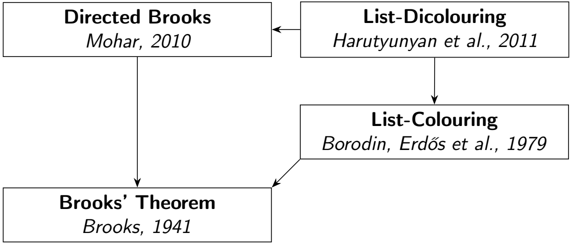

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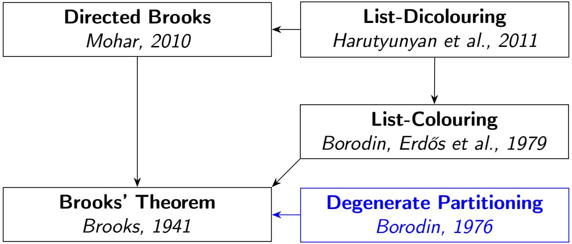

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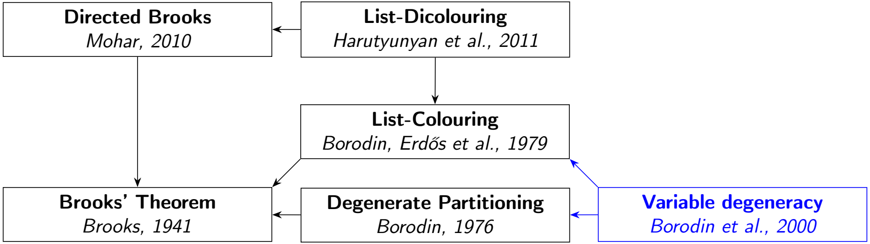

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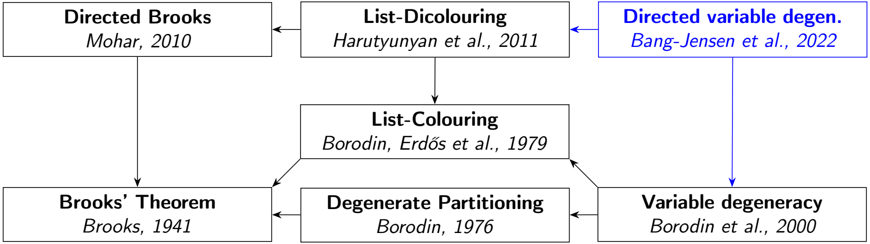

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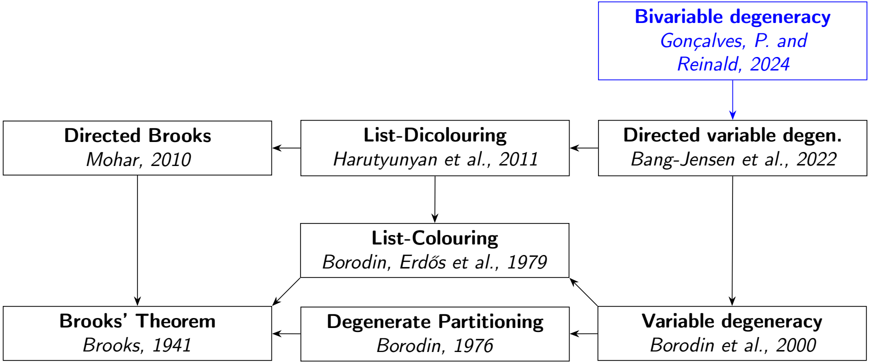

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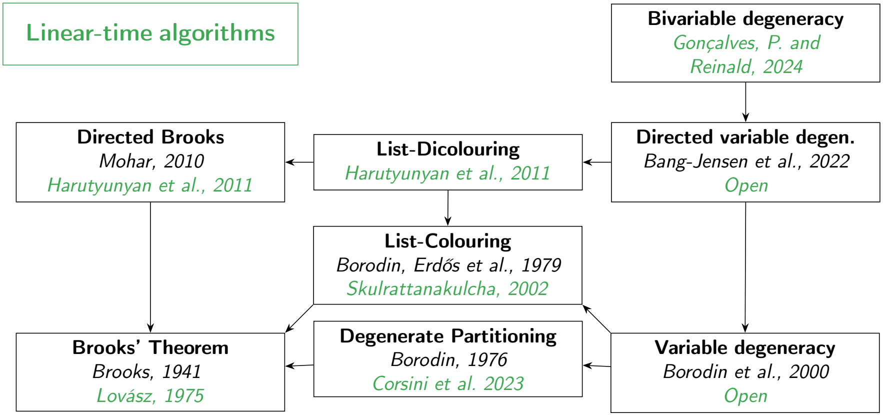

Bivariable degeneracy and F-dicolouring

A digraph D is **f-degenerate**, for $f : V(D) \to \mathbb{N}^2$, if every non-empty subdigraph H of D contains a vertex $v \in V(H)$ s.t.

$$d^-(v) < f^-(v) \quad \text{or} \quad d^+(v) < f^+(v).$$

Example: D is acyclic $\Leftrightarrow$ D is $(1,1)$-degenerate.

A digraph D is **F-dicolourable**, for $F = (f_1, \dots, f_s)$, if there exists a partition $V_1, \dots, V_s$ of $V(D)$ s.t. $D[V_i]$ is f_i-degenerate.

Examples:

- $\vec{\chi}(D) \leq s \Leftrightarrow D$ is F-dicolourable where $f_i = (1,1)$ for every $i \in [s]$.
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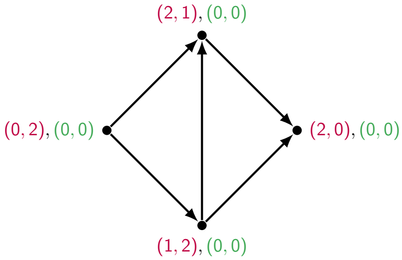
Hard pairs

(D, F) is a **hard pair** if:

- *D* **is any digraph**, and $\exists i \in [s]$, $f_i(v) = (d^-(v), d^+(v))$ and $f_j(v) = (0, 0)$ for $j \neq i$;



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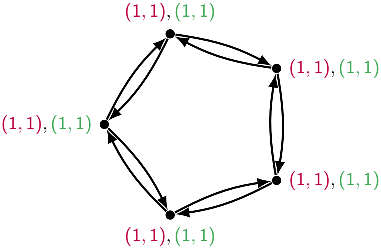

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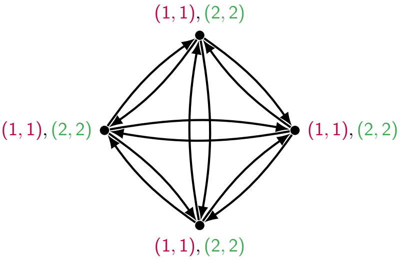

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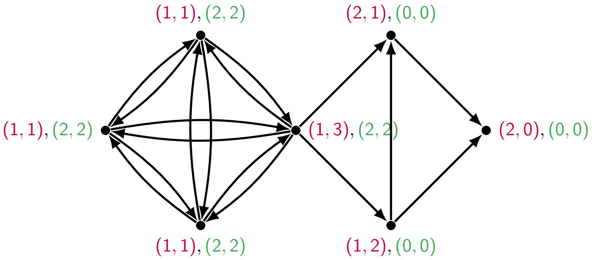
- (D, F) is obtained from two **smaller hard pairs** by gluing on a vertex.



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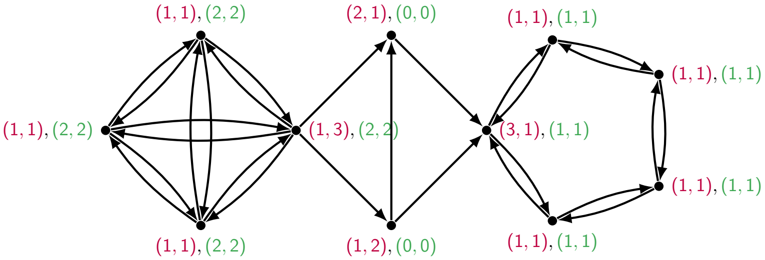

Generalisation of the Directed Brooks' Theorem via bivariable degeneracy

Theorem (Gonçalves, P., and Reinald 2024)

Let D be a connected digraph and $F = (f_1, \dots, f_s)$ be a sequence of functions $f_i \colon V(D) \to \mathbb{N}^2$ such that, for every vertex $v \in V(D)$,

$$\sum_{i=1}^{n} f_i^-(v) \geq d^-(v) \quad \text{and} \quad \sum_{i=1}^{n} f_i^+(v) \geq d^+(v).$$

*Then D is F-**dicolourable** unless (D, F) is a **hard pair**.*

Consequences:

- For symmetric functions ($f_i^- = f_i^+$) $\Rightarrow$ [Bang-Jensen et al. 2022].
- For symmetric functions valued in $\{(0,0),(1,1)\}$ $\Rightarrow$ [Harutyunyan et al. 2011].
- For functions constant equal to $(1,1)$ $\Rightarrow$ [Mohar 2010].

Generalisation of the Directed Brooks' Theorem via bivariable degeneracy

Theorem (Gonçalves, P., and Reinald 2024)

Let D be a connected digraph and $F = (f_1, \dots, f_s)$ be a sequence of functions $f_i \colon V(D) \to \mathbb{N}^2$ such that, for every vertex $v \in V(D)$,

$$\sum_{i=1}^{n} f_i^-(v) \geq d^-(v) \quad \text{and} \quad \sum_{i=1}^{n} f_i^+(v) \geq d^+(v).$$

*Then D is F-**dicolourable** unless (D, F) is a **hard pair**.*

Consequences:

- For symmetric functions $(f_i^- = f_i^+) \Rightarrow$ [Bang-Jensen et al. 2022].
- For symmetric functions valued in $\{(0,0),(1,1)\} \Rightarrow$ [Harutyunyan et al. 2011].
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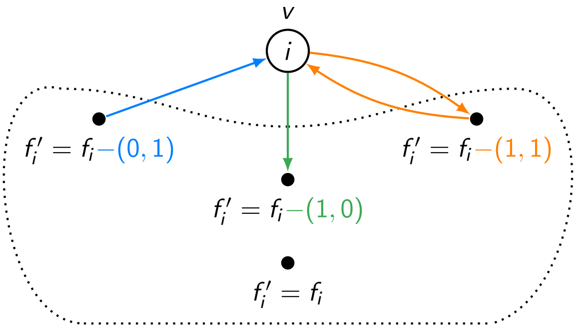
Main observation for the proof



$f_i(v) \neq (0,0)$

$f'_{j \neq i} = f_j$

$F' = (f'_1, \dots, f'_s)$

$D - v$ is F'-dicolourable $\Longrightarrow$ D is F-dicolourable.

Sketch of proof

Let $(D, F = (f_1, \ldots, f_s))$ be a **non-hard** pair .

1. Reduce to a non-hard pair $(D, (f_1', f_2'))$ by taking $f_1' = f_i$ and $f_2' = \sum_{i \neq j} f_j$.

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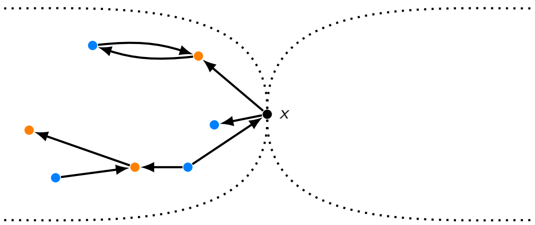

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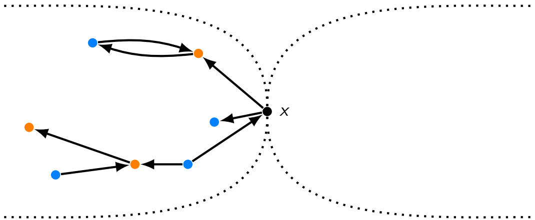

$$\tilde{f}_1(x) = \left(f_1^-(x) - |N^-(x) \cap V_1|,\ f_1^+(x) - |N^+(x) \cap V_1|\right)$$
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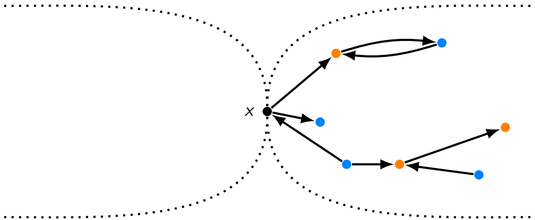

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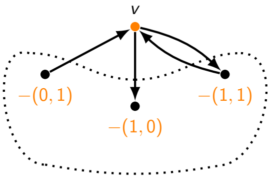

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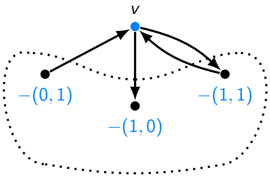

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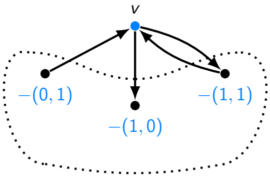

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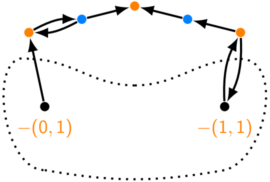

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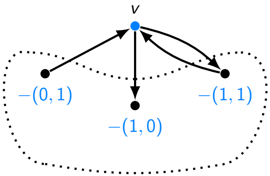

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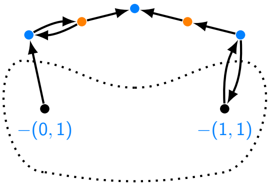

Strengthening the Directed Brooks' Theorem

$$\Delta_{\max}(D) = \max_{v \in V(D)} \max(d^-(v), d^+(v)).$$

Theorem (Mohar 2010)

Let D be a connected digraph, then $\vec{\chi}(D) \leq \Delta_{\max}(D)$ unless D is a **directed cycle**, *a* **bidirected odd cycle**, *or a* **bidirected complete graph**.

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Every digraph D satisfies $\vec{\chi}(D) \leq \Delta_{\min}(D) + 1 \leq \Delta_{\max}(D) + 1$.

Question: Can we characterise digraphs D with $\vec{\chi}(D) = \Delta_{\min}(D) + 1$?

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Theorem (Aboulker and Aubian 2022)

Deciding $\vec{\chi}(D) \leq \Delta_{\min}(D)$ is an **NP-complete** *problem.*

$\Longrightarrow$ **No easy characterisation** unless P=NP, but we can give a **necessary condition**:

Theorem (P. 2023)

Let D be a digraph with $\vec{\chi}(D) = \Delta_{\min}(D) + 1$, then:

- $\Delta_{\min}(D) \leq 1$, *or*
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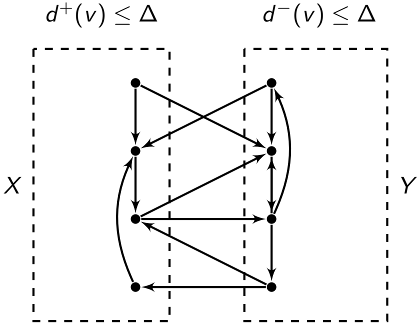

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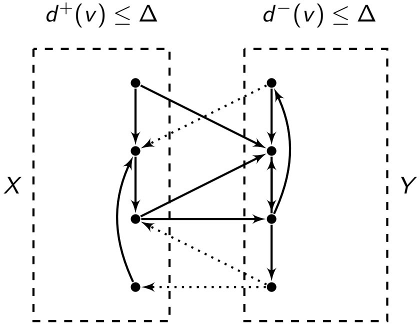

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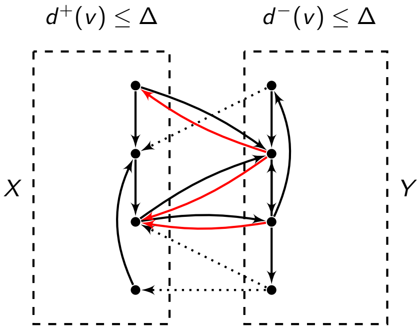

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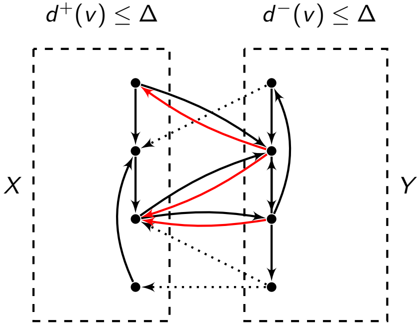

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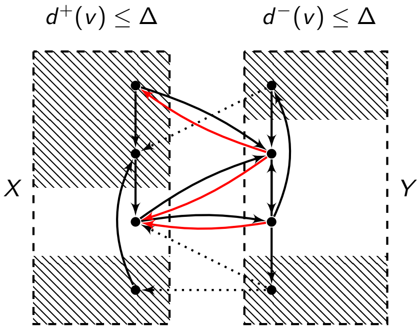

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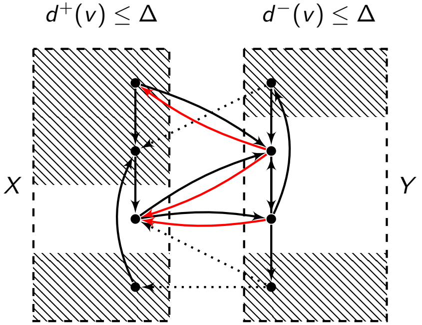
- $\vec{\chi}(H) = \Delta_{\max}(H) + 1$, the result follows from Directed Brooks' Theorem.



Further research: sublinear bounds for oriented graphs

Conjecture (Erdős and Neumann-Lara 1979)

Let $\vec{G}$ be an oriented graph, then $\vec{\chi}(\vec{G}) = O\left(\frac{\Delta_{\max}(\vec{G})}{\log \Delta_{\max}(\vec{G})}\right)$.

$\tilde{\Delta}(D) = \max_{v \in V(D)} \sqrt{d^-(v) \cdot d^+(v)}$ $\left(\text{by definition } \Delta_{\min}(D) \leq \tilde{\Delta}(D) \leq \Delta_{\max}(D)\right)$.

Theorem (Harutyunyan and Mohar 2011)

There exists $\varepsilon > 0$ s.t. every oriented graph $\vec{G}$ with $\tilde{\Delta}(\vec{G})$ large enough satisfies

$$\vec{\chi}(\vec{G}) \leq (1-\varepsilon)\tilde{\Delta}(\vec{G}).$$

Particular case: Is it true that $\vec{\chi}(K_n \square K_n) = O\left(\frac{n}{\log n}\right)$?

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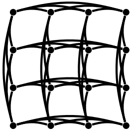

Further research: an analogue of Reed's conjecture for digraphs

Conjecture (Reed 1998): Every graph G satisfies $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

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