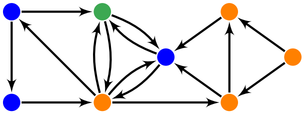
Digraph redicolouring

N. Bousquet, F. Havet, N. Nisse, L. Picasarri-Arrieta, A. Reinald

Introduction to digraph redicolouring and degeneracy

Digraph dicolouring

- **k-dicolouring** of D : partition of $V(D)$ in k parts inducing an acyclic subdigraph.



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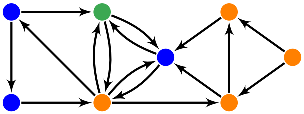
- **Dichromatic number** $\vec{\chi}(D)$: minimum k s.t. D admits a k-dicolouring.



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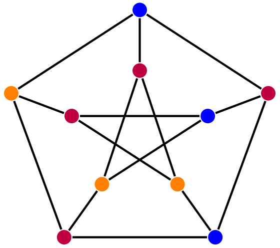

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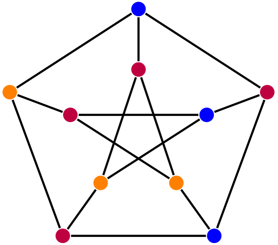

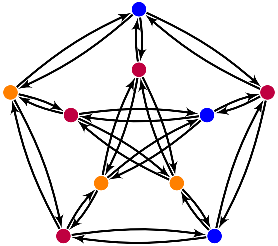

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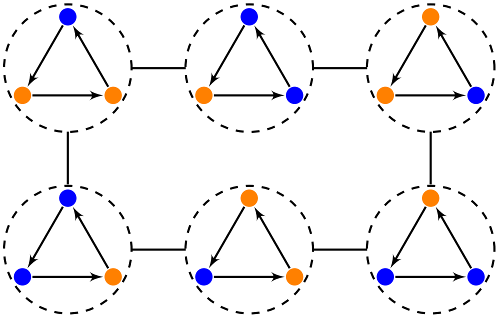
$\mathcal{D}_k(D)$: the ***k*-dicolouring graph** of D :

- $V(\mathcal{D}_k(D))$ are the k-dicolourings of D,
- $\gamma_i\gamma_j \in E(\mathcal{D}_k(D))$ if $\gamma_i = \gamma_j$ except on one vertex.



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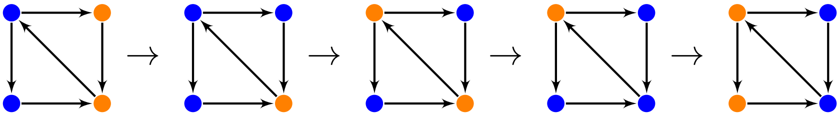
$\mathcal{G}_k(G)$: the **k-colouring graph** of G is similar.

- **recolouring sequence** : a path (or a walk) in $\mathcal{D}_k(D)$.



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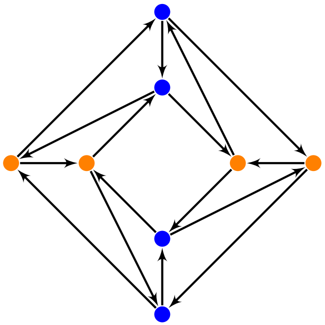
- D is ***k*-mixing** : $\mathcal{D}_k(D)$ is connected.



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$\longrightarrow$ Is D **k-mixing** ?

$\longrightarrow$ **Can we bound the diameter of $\mathcal{D}_k(D)$?**

Undirected graphs

Theorem (Bonsma et al. ; Dyer et al.)

If $k \geq \delta^*(G) + 2$, then G is k-mixing, and $diam(\mathcal{G}_k(G)) \leq 2^n - 1$.

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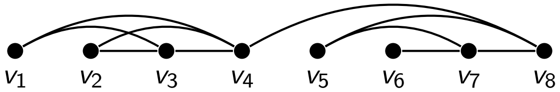

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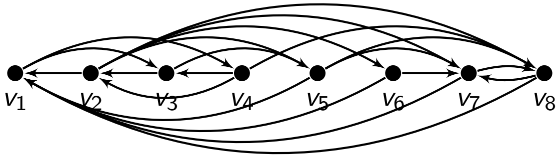

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- $\delta^*(G) = \delta^*_{\min}(\overleftrightarrow{G})$

A generalization of Cereceda's conjecture to digraphs

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Every graph G satisfies $\chi(G) \leq \delta^*(G) + 1$.

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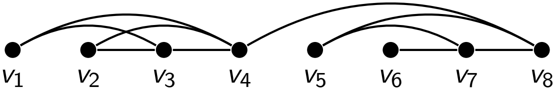

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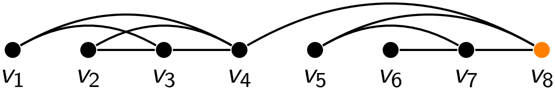

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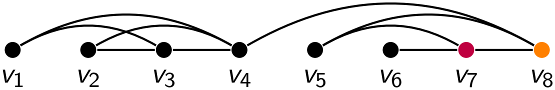

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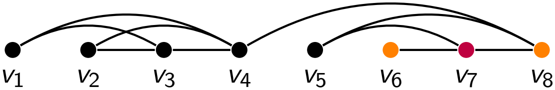

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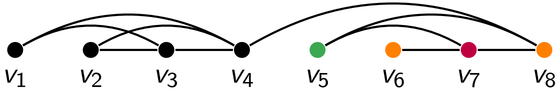

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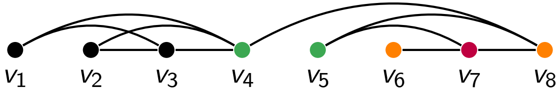

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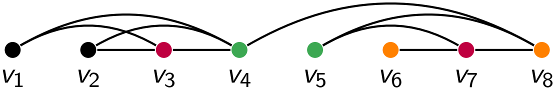

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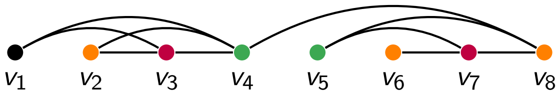

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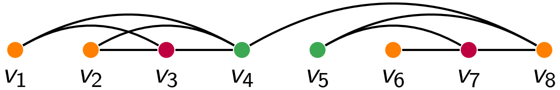

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This generalizes to the following :
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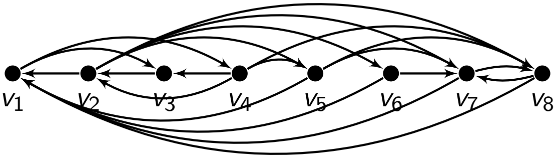

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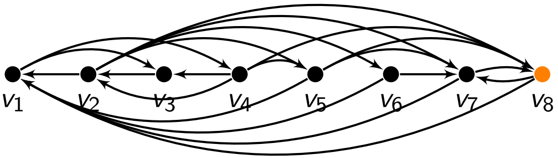

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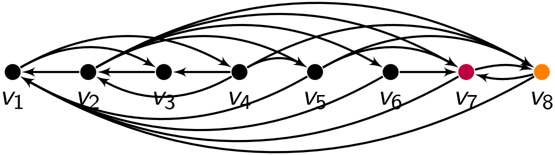

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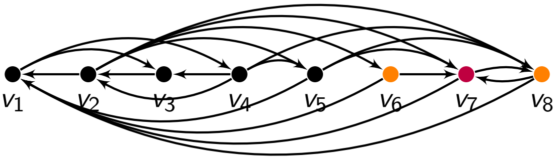

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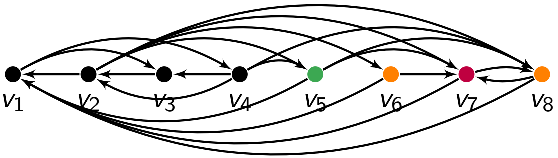

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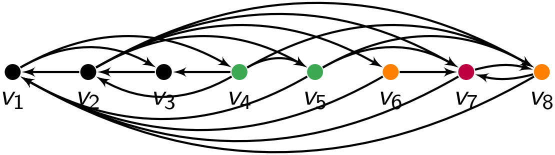

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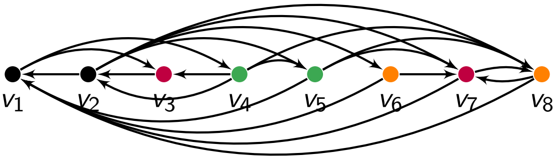

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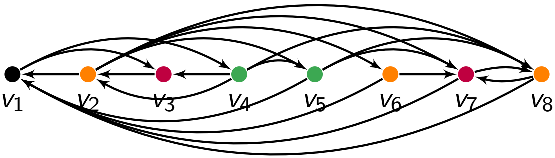

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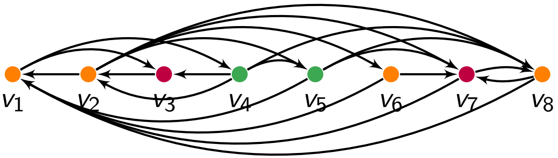

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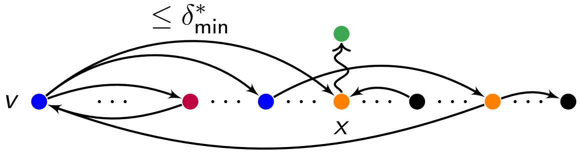

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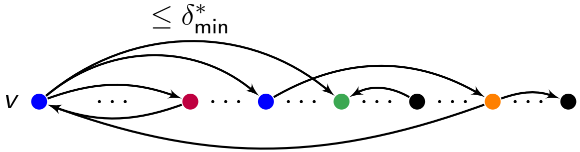

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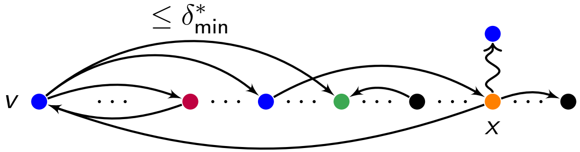

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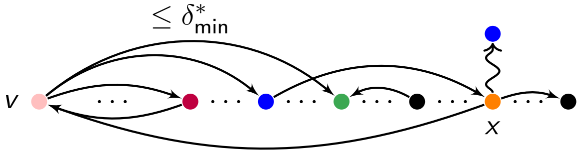

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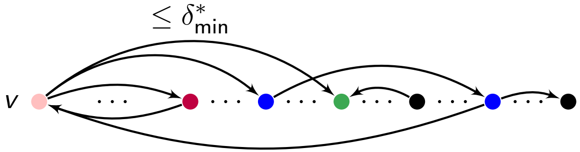

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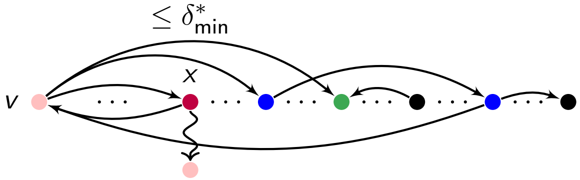

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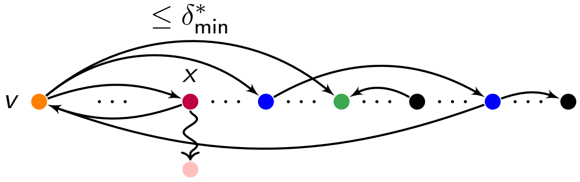

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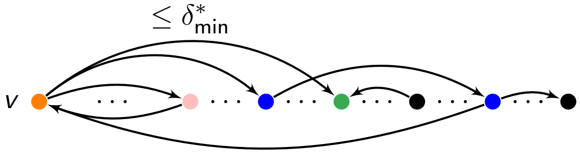

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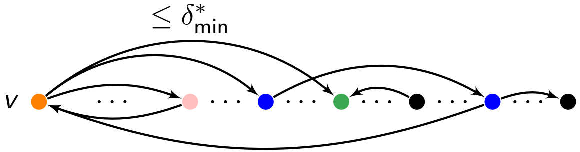

At the end we find $\alpha \longrightarrow \beta$ of length $\leq 2(2^{n-1}-1)+1 = 2^n - 1$

An analogue of Cereceda's conjecture.

Conjecture (Cereceda, 2007)

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We posed the analogue for digraphs :

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If $k \geq \delta^*_{\min}(D) + 2$, then $diam(\mathcal{D}_k(D)) = O(n^2)$.

A generalization of a result from Bousquet and Heinrich

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This generalizes to the following :

Theorem

If $k \geq \frac{3}{2}(\delta^*_{\text{min}}(D) + 1)$*, then* $diam(\mathcal{D}_k(D)) = O(n^2)$.

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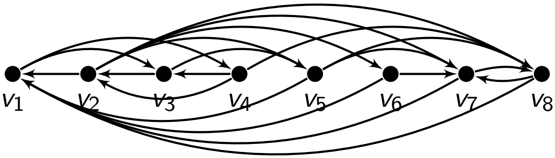

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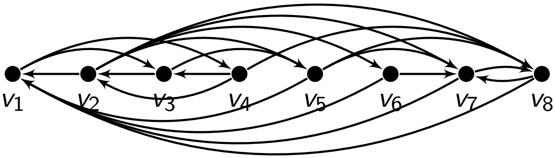

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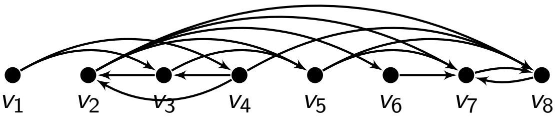

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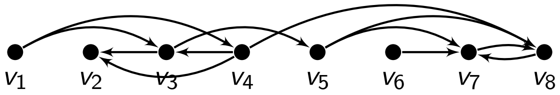

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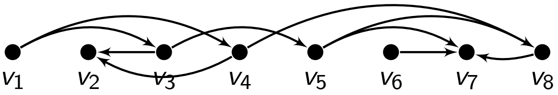

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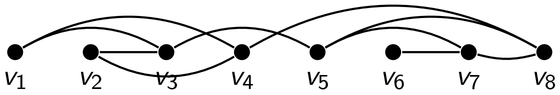

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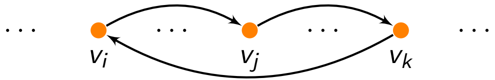

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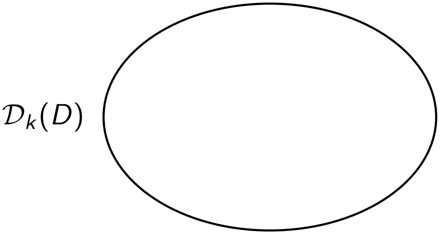

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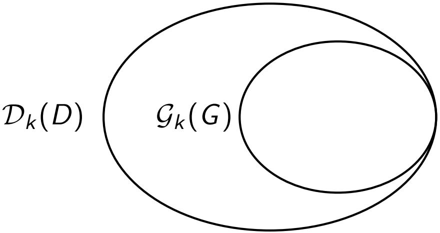

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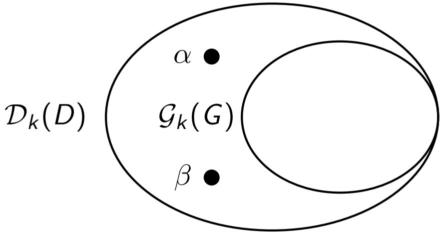

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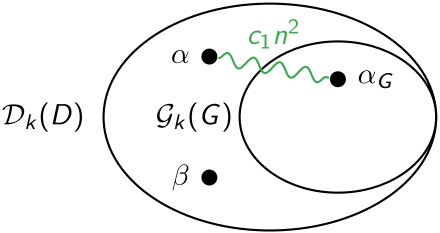

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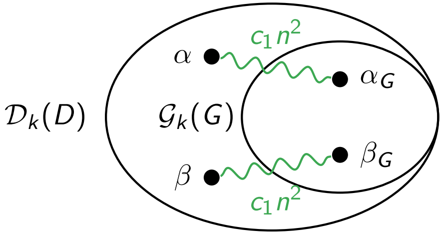

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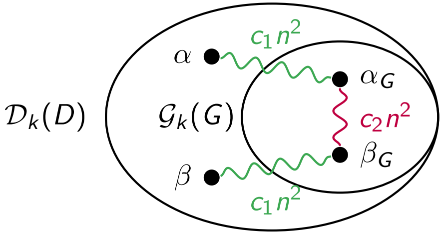

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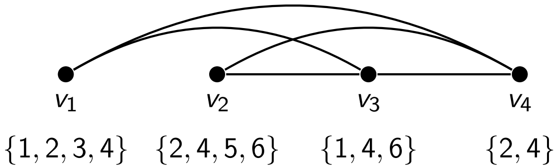

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$\Longrightarrow \forall S' \subset \{1, \ldots, k\}, |S'| = \frac{k}{3}, \exists \beta$ avoiding S', s.t. $\alpha \xrightarrow{c_3 kn} \beta$ is valid for L.

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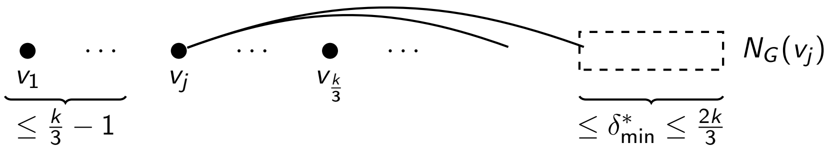
First, $\alpha \xrightarrow{\frac{k}{3}} \gamma_{\frac{k}{3}}$ where $V(H_{\frac{k}{3}})$ is well-coloured by $\gamma_{\frac{k}{3}}$:



Assume $V(H_i)$ is **well-coloured** by γ_i. Then $\gamma_i \xrightarrow{\leq 2c_3 kn + \frac{k}{3}} \gamma_{i+\frac{k}{3}}$ s.t. $V(H_{i+\frac{k}{3}})$ is **well-coloured** by $\gamma_{i+\frac{k}{3}}$:

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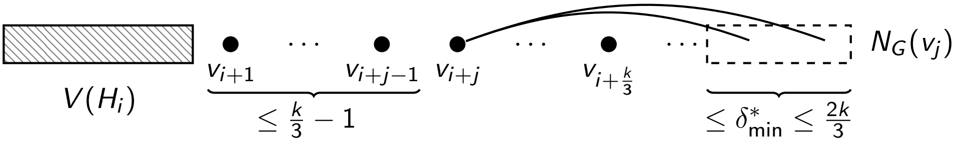

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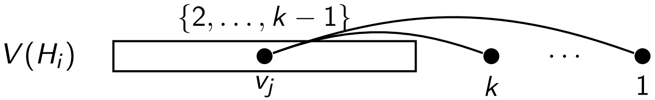

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$$|L(v_j)| \geq |N(v_j) \cap \{v_{j+1}, \dots, v_i\}| + 1 + \frac{k}{3}$$

L is **$\frac{k}{3}$-feasible** : we can apply the Lemma.

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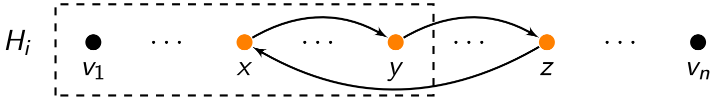
- From the Lemma : $\gamma_i \xrightarrow{\leq c_3kn} \gamma_i'$ valid for L s.t. γ_i' **avoids** S on $V(H_i)$.



Assume $V(H_i)$ is **well-coloured** by γ_i. Then $\gamma_i \xrightarrow{\leq 2c_3kn+\frac{k}{3}} \gamma_{i+\frac{k}{3}}$ s.t. $V(H_{i+\frac{k}{3}})$ is **well-coloured** by $\gamma_{i+\frac{k}{3}}$:

- Ignore $V(H_i)$, **choose** $S = c_1, \dots, c_{\frac{k}{3}}$ for $v_{i+1}, \dots, v_{i+\frac{k}{3}}$.
- Consider L the **list-assignment** of H_i where :

$$L(v_j) = \{1, \dots, k\} \setminus \gamma_i(\{N_G(v_j) \cap \{v_{i+1}, \dots, v_n\}\})$$

- From the Lemma : $\gamma_i \xrightarrow{\leq c_3kn} \gamma'_i$ valid for L s.t. γ'_i **avoids** S on $V(H_i)$.
- **Recolour** $v_{i+1}, \dots, v_{i+\frac{k}{3}}$ with S : $\gamma'_i \xrightarrow{\leq \frac{k}{3}} \tilde{\gamma}_i$, where $\tilde{\gamma}_i$ induces a colouring of $H_{i+\frac{k}{3}}$.

Assume $V(H_i)$ is **well-coloured** by γ_i. Then $\gamma_i \xrightarrow{\leq 2c_3kn+\frac{k}{3}} \gamma_{i+\frac{k}{3}}$ where $V(H_{i+\frac{k}{3}})$ is **well-coloured** by $\gamma_{i+\frac{k}{3}}$:

- $\gamma_i \xrightarrow{\leq c_3kn} \gamma_i'$.
- $\gamma_i' \xrightarrow{\leq \frac{k}{3}} \tilde{\gamma}_i$, where $\tilde{\gamma}_i$ induces a colouring of $H_{i+\frac{k}{3}}$.

Assume $V(H_i)$ is **well-coloured** by γ_i. Then $\gamma_i \xrightarrow{\leq 2c_3kn+\frac{k}{3}} \gamma_{i+\frac{k}{3}}$ where $V(H_{i+\frac{k}{3}})$ is **well-coloured** by $\gamma_{i+\frac{k}{3}}$:

- $\gamma_i \xrightarrow{\leq c_3kn} \gamma_i'$.
- $\gamma_i' \xrightarrow{\leq \frac{k}{3}} \tilde{\gamma}_i$, where $\tilde{\gamma}_i$ induces a colouring of $H_{i+\frac{k}{3}}$.
- Consider $\tilde{L}$ the **list-assignment** of H_i where :

$$\tilde{L}(v_j) = \{1,\dots,k\} \setminus \tilde{\gamma}_i(\{N_G(v_j) \cap \{v_{i+1},\dots,v_n\}\})$$

Assume $V(H_i)$ is **well-coloured** by γ_i. Then $\gamma_i \xrightarrow{\leq 2c_3kn+\frac{k}{3}} \gamma_{i+\frac{k}{3}}$ where $V(H_{i+\frac{k}{3}})$ is **well-coloured** by $\gamma_{i+\frac{k}{3}}$:

- $\gamma_i \xrightarrow{\leq c_3kn} \gamma_i'$.
- $\gamma_i' \xrightarrow{\leq \frac{k}{3}} \tilde{\gamma}_i$, where $\tilde{\gamma}_i$ induces a colouring of $H_{i+\frac{k}{3}}$.
- Consider $\tilde{L}$ the **list-assignment** of H_i where :

$$\tilde{L}(v_j) = \{1, \dots, k\} \setminus \tilde{\gamma}_i(\{N_G(v_j) \cap \{v_{i+1}, \dots, v_n\})$$

- From the Lemma : $\tilde{\gamma}_i \xrightarrow{\leq c_3kn} \gamma_{i+\frac{k}{3}}$ where $\gamma_{i+\frac{k}{3}}$ **avoids** S' (disjoint from S) on $V(H_{i+\frac{k}{3}})$.

Assume $V(H_i)$ is **well-coloured** by γ_i. Then $\gamma_i \xrightarrow{\leq 2c_3kn + \frac{k}{3}} \gamma_{i+\frac{k}{3}}$ where $V(H_{i+\frac{k}{3}})$ is **well-coloured** by $\gamma_{i+\frac{k}{3}}$:

- $\gamma_i \xrightarrow{\leq c_3kn} \gamma'_i$.
- $\gamma'_i \xrightarrow{\leq \frac{k}{3}} \tilde{\gamma}_i$.
- $\tilde{\gamma}_i \xrightarrow{\leq c_3kn} \gamma_{i+\frac{k}{3}}$.

Repeating this process at most $\frac{n}{\frac{k}{3}}$ times, we get a recolouring sequence from α to some α_G with length at most :

$$\frac{n}{\frac{k}{3}}(2c_3kn + \frac{k}{3}) \leq 6c_3n^2 + n$$

Some open problems

Using the treewidth

Theorem (Bonamy, Bousquet, 2013)

If $k \geq tw(G) + 2$, then G is k-mixing and $\mathcal{G}_k(G)$ has diameter $O(n^2)$.

Problem

If $k \geq dtw(D) + 2$, is D k-mixing ? If it is, does $\mathcal{D}_k(D)$ have a quadratic diameter ?

Using the Maximum Average Degree

Theorem

If an oriented graph D satisfies $MAD(D) < \frac{7}{2}$ then it is 2-mixing.

Conjecture

It is also true when $MAD(D) < 4$.

Using the planarity

Conjecture (Neumann-Lara)

Every oriented planar graph D has dichromatic number at most 2.

It is known that $\vec{\chi}(D) \leq 3$.

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Conjecture (Neumann-Lara)

Every oriented planar graph D has dichromatic number at most 2.

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Problem

Is every oriented planar graph D 3-mixing ?

About complexity

Theorem

For every $k \geq 2$, given a digraph D together with two k-dicolourings α,β of D, deciding if there is a recolouring sequence (with k colours) between α and β is PSPACE-complete.

Problem

What is the complexity of deciding if D is k-mixing for any fixed $k \geq 2$?

Thanks for your attention.