Recolouring Digraphs of bounded cycle-degeneracy

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Joint works: N. Bousquet, F. Havet, N. Nisse, A. Reinald, I. Sau

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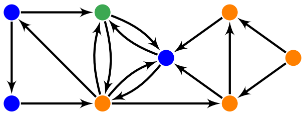
- ***k*-dicolouring** of D: partition of $V(D)$ in k parts inducing an acyclic subdigraph.



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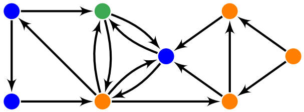

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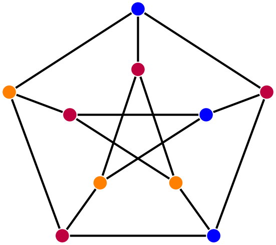

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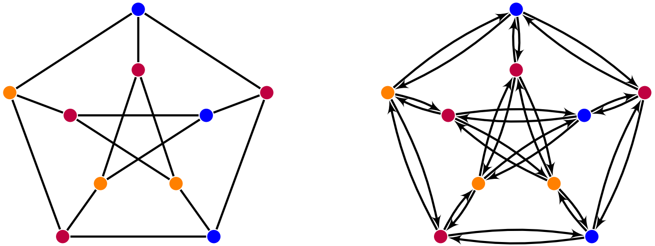

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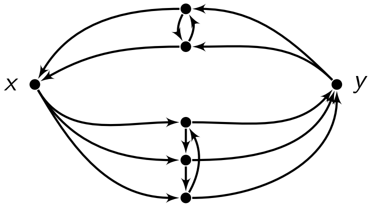

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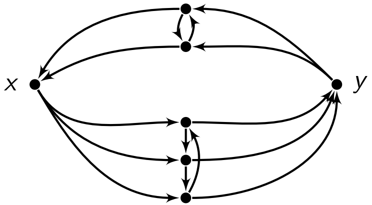

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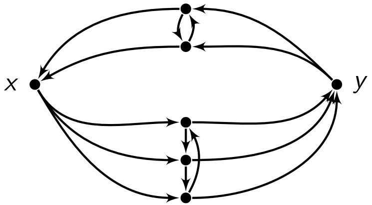

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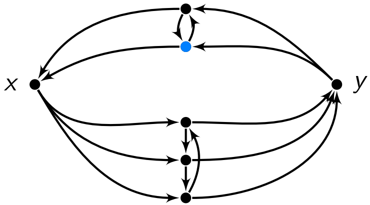

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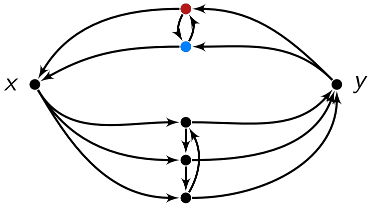

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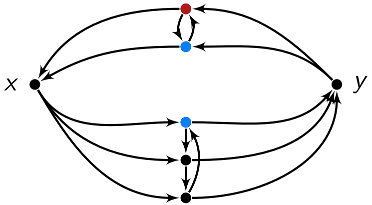

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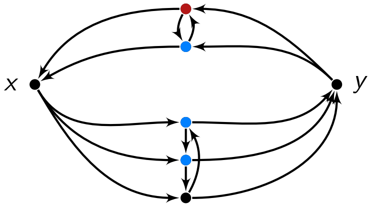

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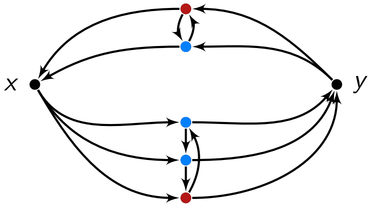

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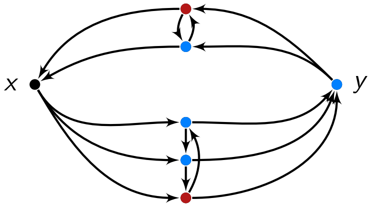

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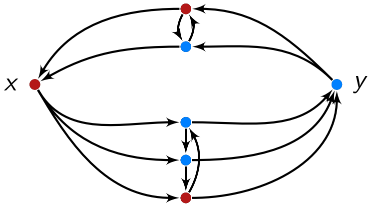
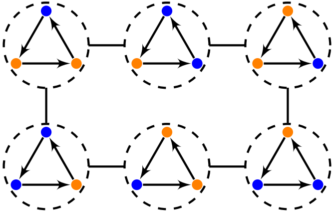


Digraph recolouring

$\mathcal{D}_k(D)$: the ***k*-dicolouring graph** of D:

- $V(\mathcal{D}_k(D))$ are the k-dicolourings of D,
- $\{\alpha, \beta\} \in E(\mathcal{D}_k(D))$ iff $\alpha = \beta$ except on exactly one vertex.



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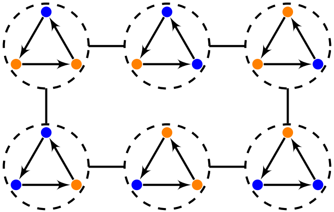
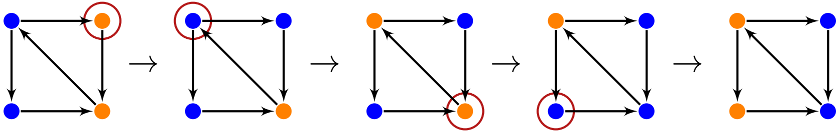


$\mathcal{C}_k(G)$: the ***k*-colouring graph** of G is the same for **proper colourings**.

- **recolouring sequence**: a path (or a walk) in $\mathcal{D}_k(D)$.



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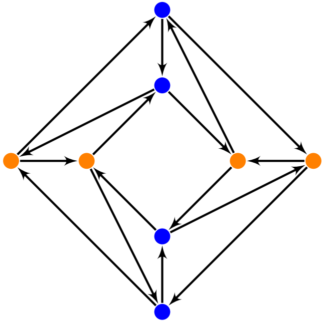

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$\longrightarrow$ **Is** D k**-mixing ?**

$\longrightarrow$ **Can we bound the diameter of** $\mathcal{D}_k(D)$ **?**

Recolouring digraphs of bounded degeneracy

Undirected graphs

Let $\delta^*(G) = \delta$,

- If $k \geq \delta + 2$, then G is k-mixing.

  (Bonsma and Cereceda '07 ; Dyer et al. '06)

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Let $\delta_{\max}^*(\vec{G}) = \delta$,

- If $k \geq \delta + 1$, then $\vec{G}$ is k-mixing.

  (Bousquet, Havet, Nisse, P., Reinald '23)

Recolouring digraphs of bounded maximum degree

Theorem (Feghali et al. '16)

G connected graph, $k \geq \Delta(G) + 1 \geq 4$, α, β proper k-colourings of G, then:

- *α or β is frozen, or*
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D connected digraph, $k \geq \Delta_{\max}(D) + 1 \geq 4$, α, β k-dicolourings of D, then:

- *α or β is frozen, or*
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Corollary

If a digraph D is not bidirected, and $k \geq \Delta_{\max}(D) + 1 \geq 4$, then $diam(\mathcal{D}_k(D)) \leq c_\Delta \cdot n^2$.

Recolouring oriented graphs (no $\overleftrightarrow{K_2}$) of bounded maximum degree

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$\vec{G}$ an oriented graph, and $k \geq \Delta_{\min}(\vec{G}) + 1$, then $diam(\mathcal{D}_k(\vec{G})) \leq 2 \cdot \Delta_{\min}(\vec{G}) \cdot n$.

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Proof mostly based on:

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Every oriented graph $\vec{G}$ with $\Delta_{\min}(\vec{G}) \geq 2$ satisfies $\vec{\chi}(\vec{G}) \leq \Delta_{\min}(\vec{G})$.

A linear bound for graphs of bounded maximum degree

Theorem (Bousquet et al. '22)

G connected graph, $k \geq \Delta(G) + 1 \geq 4$, α, β proper k-colourings of G, then:

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Question: Analogue for digraphs?

Recolouring digraphs of bounded treewidth

For every graph G, $\text{tw}(G) \geq \delta^*(G)$.

Theorem (Bonamy and Bousquet '18)

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Question: Analogues for directed-Treewidth, Kelly-width or DAG-width ?

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$$\mathrm{Mad}(G) = \max_{H \subseteq G} \left(\frac{1}{|V(H)|} \sum_{v \in V(H)} d^H(v) \right) \geq \delta^*(G)$$

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Theorem (Feghali '21)

Let G be a graph and $\varepsilon > 0$ such that $Mad(G) = d - \varepsilon$. For every $k \geq d + 1$, $diam(\mathcal{C}_k) \leq c_{d,\varepsilon} \cdot n \cdot \log^{d-1}(n)$.

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Conjecture (Bousquet, Havet, Nisse, P., Reinald)

Let $\vec{G}$ be an oriented graph that is not k-mixing, $Mad(G) \geq 2k$.

Open for every $k \geq 2$.

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Question: Is every oriented planar graph 3-mixing ?

Complexity

Theorem (Bousquet, Havet, Nisse, P., Reinald)

Given D a digraph, α, β k-dicolourings of D, deciding if there is a recolouring sequence between α and β is PSPACE-complete for every fixed $k \geq 2$.

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What is the complexity of deciding whether a digraph is k-mixing (for any fixed $k \geq 2$) ?

Open for every fixed $k \geq 4$ in the undirected case.

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Thank you!