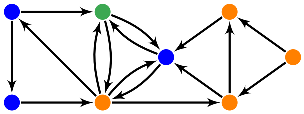
Recolouring digraphs with bounded maximum degree

Lucas Picasarri-Arrieta

Séminaire LaBRI

Digraph dicolouring

- ***k*-dicolouring** of D: partition of $V(D)$ in k parts inducing an acyclic subdigraph.



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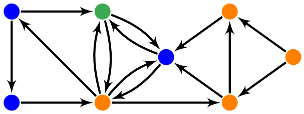

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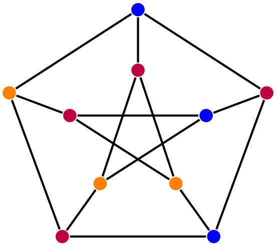

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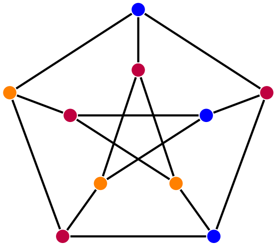

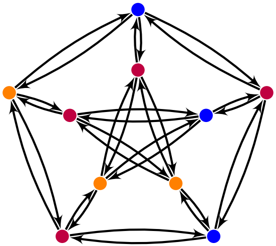

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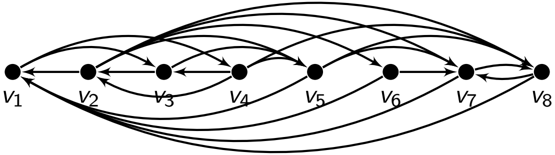
- $\vec{\chi}(D) \leq \delta^*_{\min}(D) + 1$.

An easy bound on the dichromatic number

Every digraph D satisfies $\vec{\chi}(D) \leq \delta^*_{\min}(D) + 1$.



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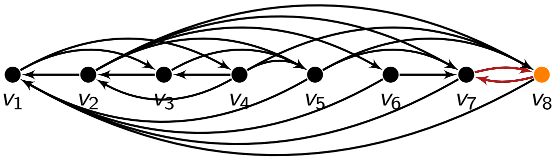

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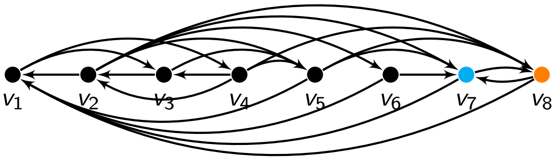

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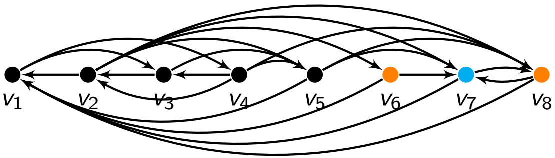

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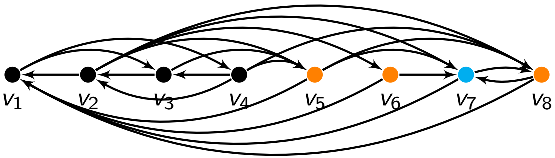

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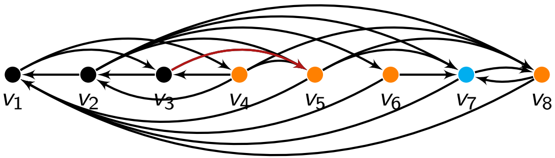

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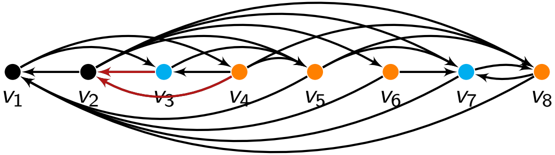

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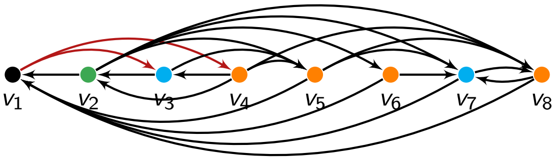

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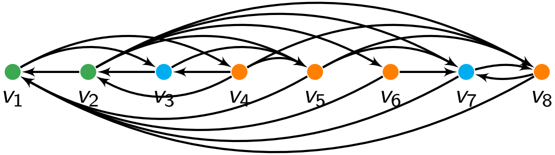
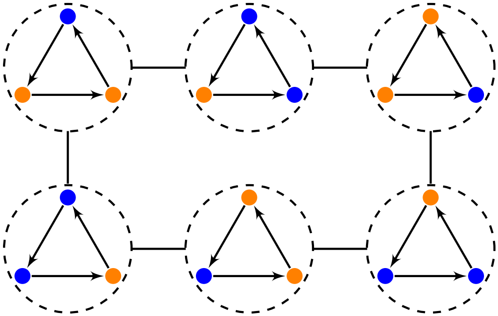


Digraph recolouring

$\mathcal{D}_k(D)$: the ***k*-dicolouring graph** of D:

- $V(\mathcal{D}_k(D))$ are the k-dicolourings of D,
- $\gamma_i\gamma_j \in E(\mathcal{D}_k(D))$ if $\gamma_i = \gamma_j$ except on one vertex.



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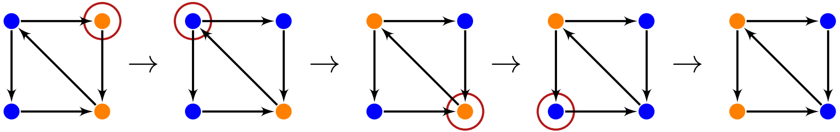
$\mathcal{C}_k(G)$: the ***k*-colouring graph** of G is similar.

- **recolouring sequence**: a path (or a walk) in $\mathcal{D}_k(D)$.



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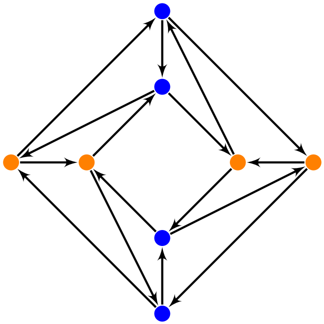

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$\longrightarrow$ **Is D k-mixing ?**

$\longrightarrow$ **Can we bound the diameter of $\mathcal{D}_k(D)$?**

Recolouring digraphs of bounded degeneracy

Undirected graphs

Let $\delta^*(G) = d$ and $k \geq d+2$,

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Let $\delta^*_{\max}(\vec{G}) = d$,

- If $k \geq d+1$, then $\vec{G}$ is k-mixing. [BHNPR]

Recolouring digraphs of bounded maximum degree

Theorem (Feghali et al. '16)

G a connected graph with $\Delta(G) = \Delta \geq 3$, $k \geq \Delta + 1$, α, β: k-colourings of G, then:

- *α is frozen, or*
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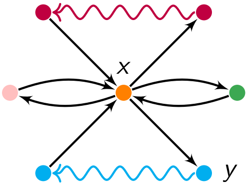
If D is not bidirected, and $k \geq \Delta_{\max}(D) + 1 \geq 4$, diameter$(\mathcal{D}_k(D)) \leq c_\Delta n^2$.



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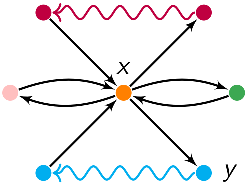

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If D is not bidirected, and $k \geq \Delta_{\max}(D) + 1 \geq 4$, $\mathrm{diameter}(\mathcal{D}_k(D)) \leq c_\Delta n^2$.

We can do better on oriented graphs: *diameter* $\leq 2\Delta_{\min}(\vec{G})n$ when $k \geq \Delta_{\min}(\vec{G}) + 1$.

Strengthening Brooks' Theorem for oriented graphs

Theorem

If $\vec{\chi}(D) = \Delta_{\min}(D) + 1 = \Delta + 1$, *then:*

- $\Delta = 1$, *or*
- $\Delta = 2$ *and* D *contains* $\overleftrightarrow{K_2}$, *or*
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Strengthening Brooks' Theorem for oriented graphs

Theorem

If $\vec{\chi}(D) = \Delta_{\min}(D) + 1 = \Delta + 1$, *then:*

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Corollary

If $\Delta_{\min}(\vec{G}) \geq 2$, *then* $\vec{\chi}(\vec{G}) \leq \Delta_{\min}(\vec{G})$.

Strengthening Brooks' Theorem for oriented graphs

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If $\Delta_{\min}(\vec{G}) \geq 2$*, then* $\vec{\chi}(\vec{G}) \leq \Delta_{\min}(\vec{G})$.

Theorem (Harutyunyan and Mohar '11)

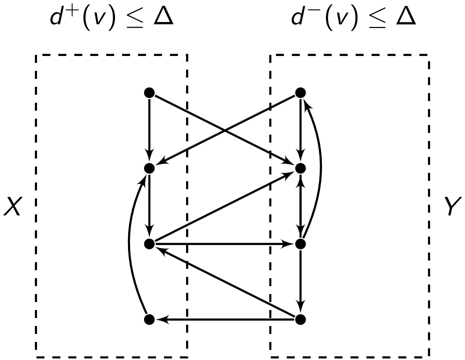
If $\vec{\chi}(D) = \Delta_{\max}(D) + 1 = \Delta + 1$*, then:*

- D *is a directed cycle, or*
- D *is a bidirected odd cycle, or*
- D *is* $\overleftrightarrow{K_{\Delta+1}}$.

Strengthening Brooks' Theorem for oriented graphs

Proof

- Assume $\vec{\chi}(D) = \Delta_{\min}(D) + 1 = \Delta + 1 \geq 3$.



Strengthening Brooks' Theorem for oriented graphs

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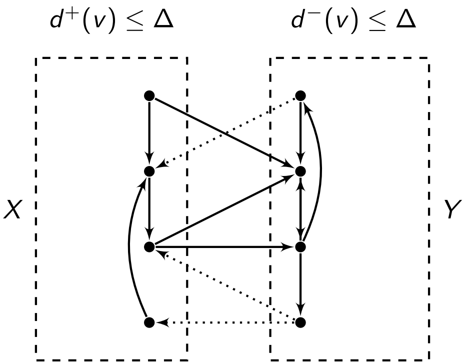
- Assume $\vec{\chi}(D) = \Delta_{\min}(D) + 1 = \Delta + 1 \geq 3$.
- Remove all arcs from Y to X.



Strengthening Brooks' Theorem for oriented graphs

Proof

- Assume $\vec{\chi}(D) = \Delta_{\min}(D) + 1 = \Delta + 1 \geq 3$.
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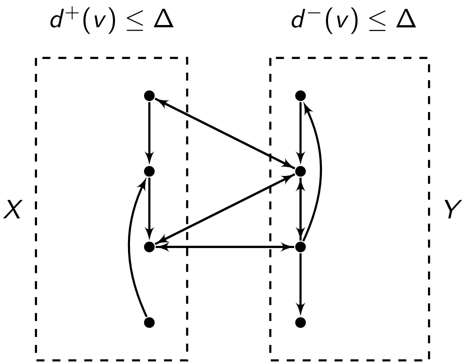

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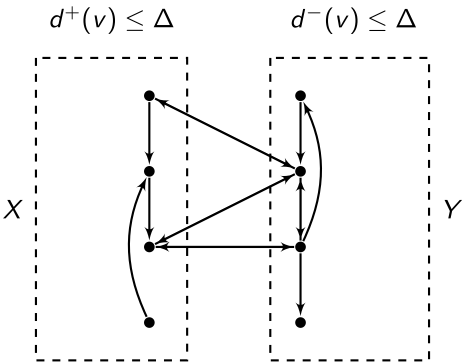

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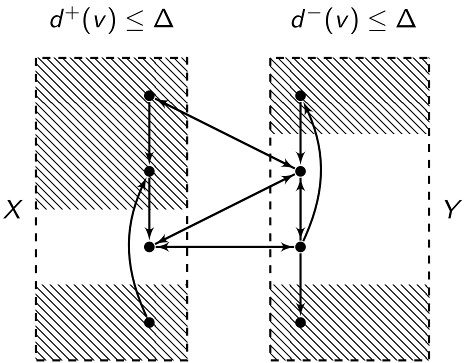
- Let $H \subseteq D'$ be $(\Delta + 1)$-dicritical. It must be Δ-diregular since:

$$\sum_{x \in X_H} d_H^+(x) = \Delta |X_H| = \sum_{x \in X_H} d_H^-(x)$$



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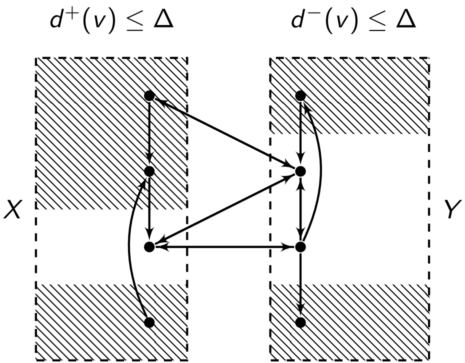
- $\vec{\chi}(H) = \Delta_{\max}(H) + 1$, the result follows from Directed Brooks' Theorem.



Recolouring oriented graphs of maximum degree 1

Theorem

If $\Delta_{\min}(\vec{G}) = 1$*, then* $\mathcal{D}_2(\vec{G})$ *has diameter* n*.*

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Sketch of the proof

- $\vec{G}$ is not $\vec{C}_n$ and is strongly connected.

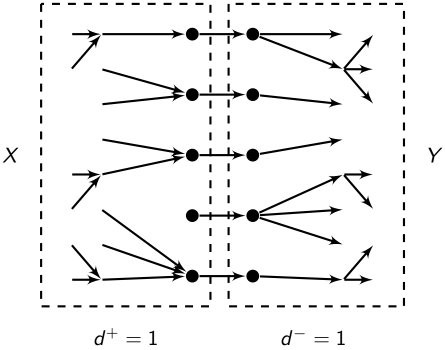
Recolouring oriented graphs of maximum degree 1

Theorem

If $\Delta_{\min}(\vec{G}) = 1$, then $\mathcal{D}_2(\vec{G})$ has diameter n.

Sketch of the proof

- $\vec{G}$ is not $\vec{C}_n$ and is strongly connected.
- $\vec{G}[X]$: disjoint union of in-trees.
- $\vec{G}[Y]$: disjoint union of out-trees.
- $A(X, Y)$ is a perfect matching between the roots.



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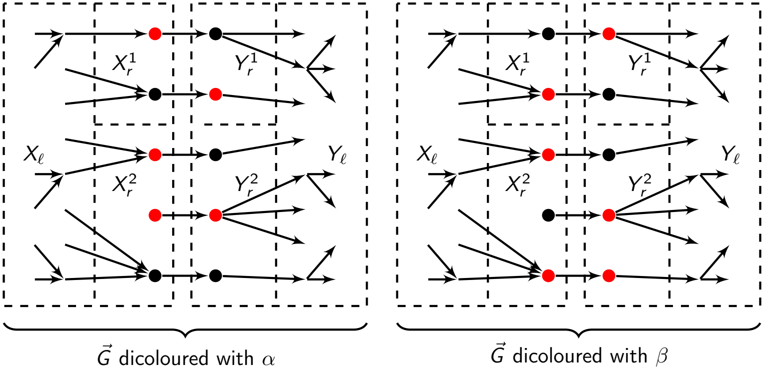

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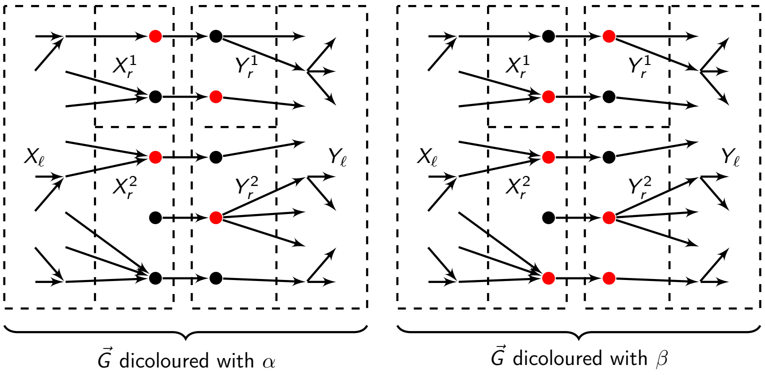

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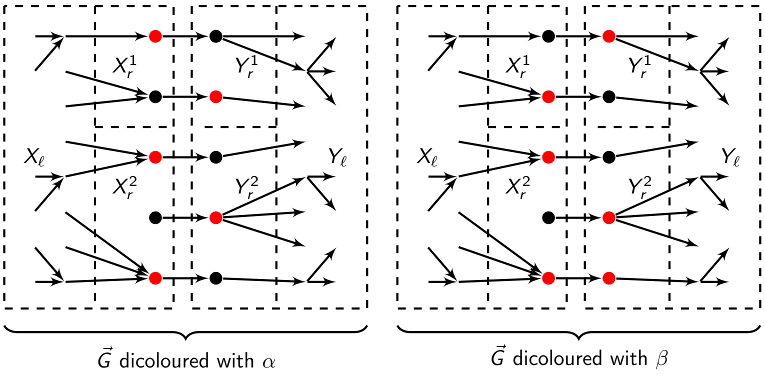

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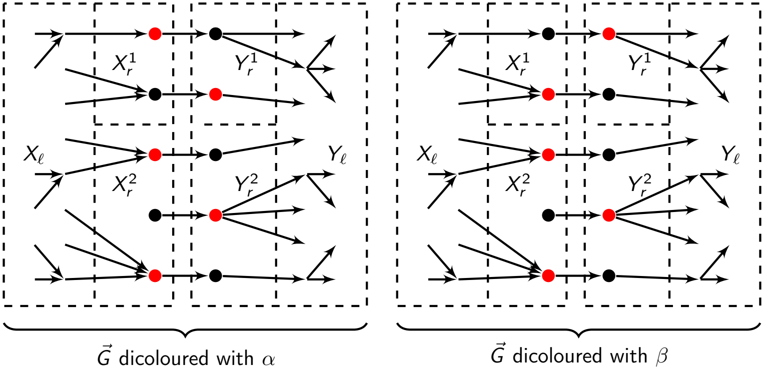

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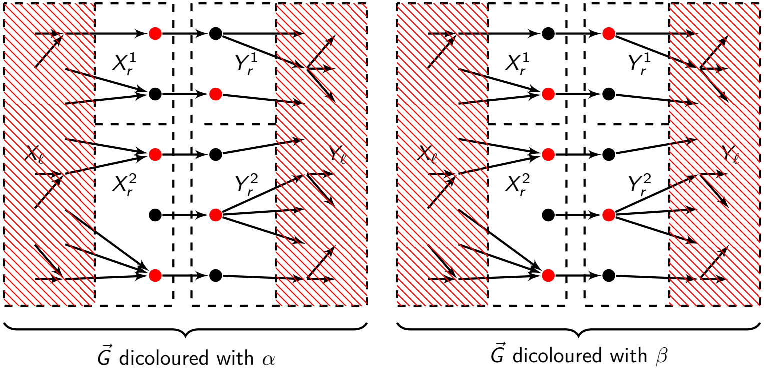

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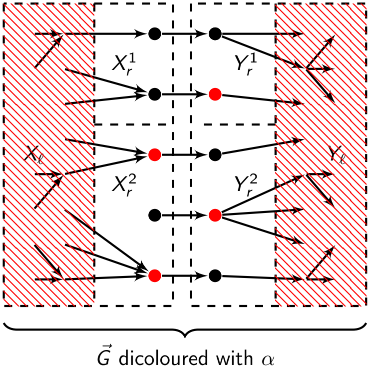

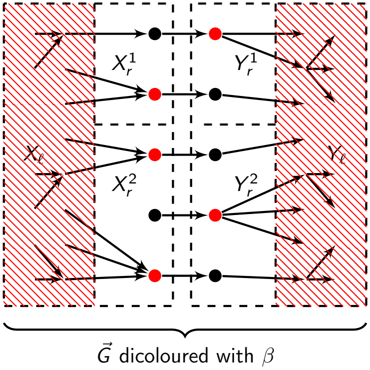

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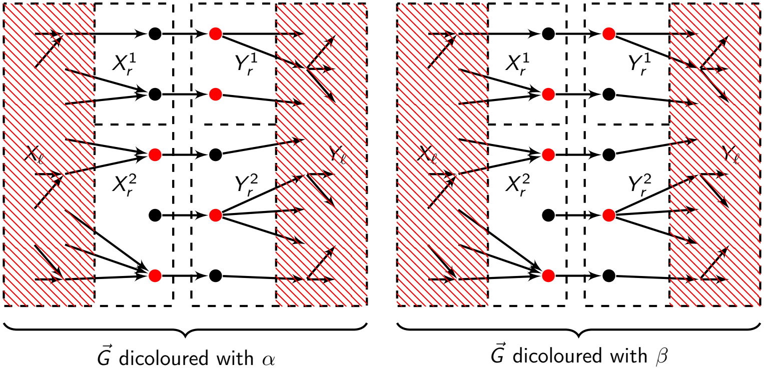

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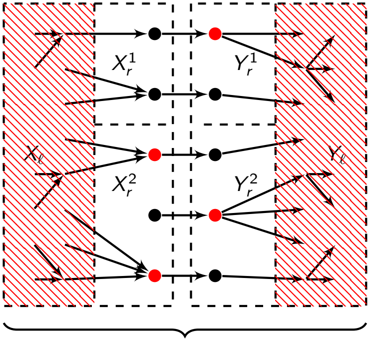
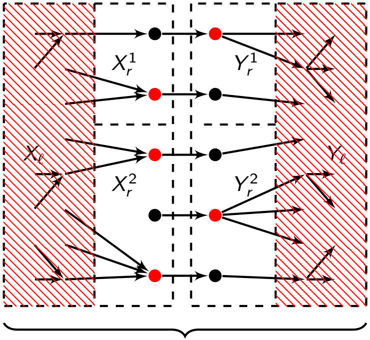
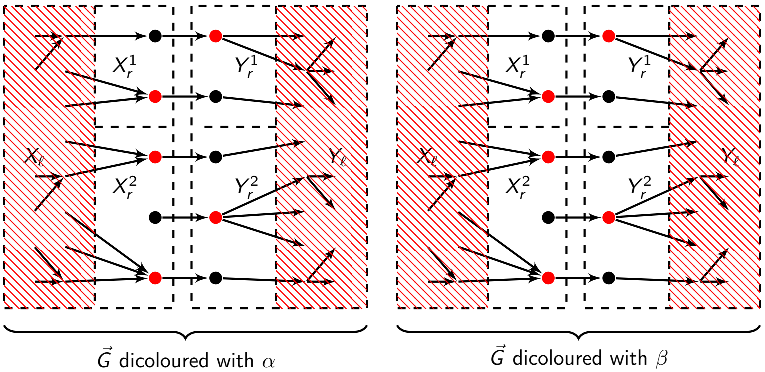


$\vec{G}$ dicoloured with α



$\vec{G}$ dicoloured with β

Recolouring oriented graphs of maximum degree 1



Recolouring oriented graphs of bounded maximum degree

Theorem

If $k \geq \Delta_{\min}(\vec{G}) + 1$*, then diameter*$(\mathcal{D}_k(\vec{G})) \leq 2\Delta_{\min}(\vec{G})n$.

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- Let α, β be two k-dicolourings.

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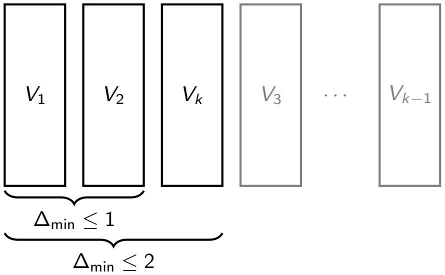
- By the previous results, $\alpha \xrightarrow{2n} \alpha'$, where α' is a $(k-1)$-dicolouring.



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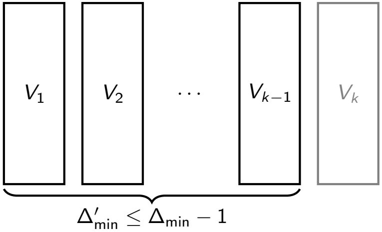
- By induction on $\Delta_{\min}(\vec{G})$, $\beta \xrightarrow{2(\Delta-1)n} \alpha'$.



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Question

Is there an absolute constant c such that $diameter(\mathcal{D}_k(\vec{G})) \leq cn$?

Open questions

Theorem (Bousquet, Havet, Nisse, P. and Reinald)

Every oriented graph $\vec{G}$ with $Mad(\vec{G}) < \frac{7}{2}$ is 2-mixing.

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Thank you !