$(\Delta - 1)$-dicolouring of digraphs

A directed analogue of Borodin–Kostochka's Conjecture

A. Harutyunyan[1], K. Kawarabayashi[2], L. Picasarri-Arrieta[2], G. Puig i Surroca[1]

09/09/2025

[1]LAMSADE, Université Paris Dauphine - PSL, Paris, France.
[2]National Institute of Informatics, Tokyo, Japan.

Definition

- $\omega(G)$: **clique number** of G

- $\Delta(G)$: **maximum degree** of G

- $\chi(G)$: **chromatic number** of G

Proposition: Every graph G satisfies $\omega(G) \leqslant \chi(G) \leqslant \Delta(G) + 1$.

Question: Does χ being close to $\Delta + 1$ implies that ω is close to χ?

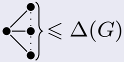
Theorem (Brooks, 1941)

For every graph G, if $\chi(G) = \Delta(G) + 1$ and $\Delta(G) \geqslant 3$ then $\omega(G) = \Delta(G) + 1$.

Definition

- $\omega(G)$: **clique number** of G

- $\Delta(G)$: **maximum degree** of G



- $\chi(G)$: **chromatic number** of G

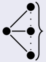
Proposition: Every graph G satisfies $\omega(G) \leqslant \chi(G) \leqslant \Delta(G) + 1$.

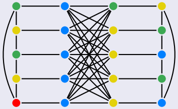
Question: Does χ being close to $\Delta + 1$ implies that ω is close to χ?

Theorem (Brooks, 1941)

For every graph G, if $\chi(G) = \Delta(G) + 1$ and $\Delta(G) \geqslant 3$ then $\omega(G) = \Delta(G) + 1$.

Definition

- $\omega(G)$: **clique number** of G 
- $\Delta(G)$: **maximum degree** of G 

- $\chi(G)$: **chromatic number** of G 

Proposition: Every graph G satisfies $\omega(G) \leqslant \chi(G) \leqslant \Delta(G) + 1$.

Question: Does χ being close to $\Delta + 1$ implies that ω is close to χ?

Theorem (Brooks, 1941)

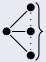
For every graph G, if $\chi(G) = \Delta(G) + 1$ and $\Delta(G) \geqslant 3$ then $\omega(G) = \Delta(G) + 1$.

ω, Δ, and χ

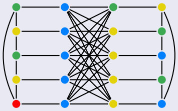
Definition

- $\omega(G)$: **clique number** of G

- $\Delta(G)$: **maximum degree** of G



- $\chi(G)$: **chromatic number** of G



Proposition: Every graph G satisfies $\omega(G) \leqslant \chi(G) \leqslant \Delta(G) + 1$.

Question: Does χ being close to $\Delta + 1$ implies that ω is close to χ?

Theorem (Brooks, 1941)

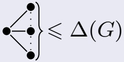
For every graph G, if $\chi(G) = \Delta(G) + 1$ and $\Delta(G) \geqslant 3$ then $\omega(G) = \Delta(G) + 1$.

ω, Δ, and χ

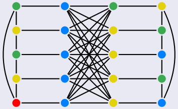
Definition

- $\omega(G)$: **clique number** of G

- $\Delta(G)$: **maximum degree** of G



- $\chi(G)$: **chromatic number** of G



Proposition: Every graph G satisfies $\omega(G) \leqslant \chi(G) \leqslant \Delta(G) + 1$.

Question: Does χ being close to $\Delta + 1$ implies that ω is close to χ?

Theorem (Brooks, 1941)

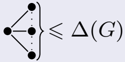
For every graph G, if $\chi(G) = \Delta(G) + 1$ and $\Delta(G) \geqslant 3$ then $\omega(G) = \Delta(G) + 1$.

ω, Δ, and χ

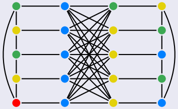
Definition

- $\omega(G)$: **clique number** of G

- $\Delta(G)$: **maximum degree** of G



- $\chi(G)$: **chromatic number** of G



Proposition: Every graph G satisfies $\omega(G) \leqslant \chi(G) \leqslant \Delta(G) + 1$.

Question: Does χ being close to $\Delta + 1$ implies that ω is close to χ?

Theorem (Brooks, 1941)

For every graph G, if $\chi(G) = \Delta(G) + 1$ and $\Delta(G) \geqslant 3$ then $\omega(G) = \Delta(G) + 1$.

Conjecture (Borodin and Kostochka, 1977)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 9$ then $\omega(G) \geqslant \Delta(G)$.

Remark: It is necessary to take $\Delta(G) \geqslant 9$.

$\Delta = 8$

$\omega = 6$

$\chi = 8$

First results:

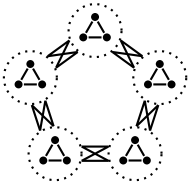
- **Borodin and Kostochka, 1977:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \frac{1}{2}(\Delta(G)+1)$.
- **Kostochka, 1980:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 28$.
- **Mozhan, 1987:** If $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 31$ then $\omega(G) \geqslant \Delta(G) - 3$.

Borodin–Kostochka's Conjecture

Conjecture (Borodin and Kostochka, 1977)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 9$ then $\omega(G) \geqslant \Delta(G)$.

Remark: It is necessary to take $\Delta(G) \geqslant 9$.



$\Delta = 8$

$\omega = 6$

$\chi = 8$

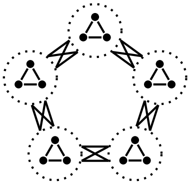
First results:

- **Borodin and Kostochka, 1977:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \frac{1}{2}(\Delta(G) + 1)$.
- **Kostochka, 1980:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 28$.
- **Mozhan, 1987:** If $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 31$ then $\omega(G) \geqslant \Delta(G) - 3$.

Conjecture (Borodin and Kostochka, 1977)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 9$ then $\omega(G) \geqslant \Delta(G)$.

Remark: It is necessary to take $\Delta(G) \geqslant 9$.



$\Delta = 8$

$\omega = 6$

$\chi = 8$

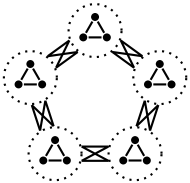
First results:

- **Borodin and Kostochka, 1977:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \frac{1}{2}(\Delta(G) + 1)$.
- **Kostochka, 1980:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 28$.
- **Mozhan, 1987:** If $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 31$ then $\omega(G) \geqslant \Delta(G) - 3$.

Conjecture (Borodin and Kostochka, 1977)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 9$ then $\omega(G) \geqslant \Delta(G)$.

Remark: It is necessary to take $\Delta(G) \geqslant 9$.



$\Delta = 8$

$\omega = 6$

$\chi = 8$

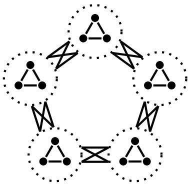
First results:

- **Borodin and Kostochka, 1977:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \frac{1}{2}(\Delta(G) + 1)$.
- **Kostochka, 1980:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 28$.
- **Mozhan, 1987:** If $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 31$ then $\omega(G) \geqslant \Delta(G) - 3$.

Conjecture (Borodin and Kostochka, 1977)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 9$ then $\omega(G) \geqslant \Delta(G)$.

Remark: It is necessary to take $\Delta(G) \geqslant 9$.



$\Delta = 8$

$\omega = 6$

$\chi = 8$

First results:

- **Borodin and Kostochka, 1977:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \frac{1}{2}(\Delta(G) + 1)$.
- **Kostochka, 1980:** If $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 28$.
- **Mozhan, 1987:** If $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 31$ then $\omega(G) \geqslant \Delta(G) - 3$.

Reed's conjecture

Conjecture (Reed, 1998)

Every graph G satisfies $\chi(G) \leqslant \left\lceil \frac{1}{2}(\Delta(G)+1) + \frac{1}{2}\omega(G) \right\rceil$.

Consequence: If $\chi(G) \geqslant \Delta(G)+1-c$ then $\omega(G) \geqslant \Delta(G)-2c$.

Theorem (Reed, 1998)

There exists $\varepsilon > 0$ *s.t. every graph G satisfies* $\chi(G) \leqslant \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.

Theorem (Reed, 1998)

For every graph G, if $\Delta(G) \geqslant \Delta_c$ *and* $\chi(G) \geqslant \Delta(G)+1-c$, *then* $\omega(G) \geqslant \Delta(G)-2c$.

Consequence: If $\Delta(G)$ is large enough and $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G)-2$.

Theorem (Reed, 1999)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ *and* $\Delta(G) \geqslant 10^{14}$ *then* $\omega(G) \geqslant \Delta(G)$.

Reed's conjecture

Conjecture (Reed, 1998)

Every graph G satisfies $\chi(G) \leqslant \left\lceil \frac{1}{2}(\Delta(G)+1) + \frac{1}{2}\omega(G) \right\rceil$.

Consequence: If $\chi(G) \geqslant \Delta(G) + 1 - c$ then $\omega(G) \geqslant \Delta(G) - 2c$.

Theorem (Reed, 1998)

There exists $\varepsilon > 0$ *s.t. every graph G satisfies* $\chi(G) \leqslant \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.

Theorem (Reed, 1998)

For every graph G, if $\Delta(G) \geqslant \Delta_c$ *and* $\chi(G) \geqslant \Delta(G) + 1 - c$*, then* $\omega(G) \geqslant \Delta(G) - 2c$.

Consequence: If $\Delta(G)$ is large enough and $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 2$.

Theorem (Reed, 1999)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ *and* $\Delta(G) \geqslant 10^{14}$ *then* $\omega(G) \geqslant \Delta(G)$.

Reed's conjecture

Conjecture (Reed, 1998)

Every graph G satisfies $\chi(G) \leqslant \left\lceil \frac{1}{2}(\Delta(G)+1) + \frac{1}{2}\omega(G) \right\rceil$.

Consequence: If $\chi(G) \geqslant \Delta(G) + 1 - c$ then $\omega(G) \geqslant \Delta(G) - 2c$.

Theorem (Reed, 1998)

There exists $\varepsilon > 0$ *s.t. every graph* G *satisfies* $\chi(G) \leqslant \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.

Theorem (Reed, 1998)

For every graph G*, if* $\Delta(G) \geqslant \Delta_c$ *and* $\chi(G) \geqslant \Delta(G) + 1 - c$*, then* $\omega(G) \geqslant \Delta(G) - 2c$.

Consequence: If $\Delta(G)$ is large enough and $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 2$.

Theorem (Reed, 1999)

For every graph G*, if* $\chi(G) \geqslant \Delta(G)$ *and* $\Delta(G) \geqslant 10^{14}$ *then* $\omega(G) \geqslant \Delta(G)$.

Reed's conjecture

Conjecture (Reed, 1998)

Every graph G satisfies $\chi(G) \leqslant \left\lceil \frac{1}{2}(\Delta(G)+1) + \frac{1}{2}\omega(G) \right\rceil$.

Consequence: If $\chi(G) \geqslant \Delta(G) + 1 - c$ then $\omega(G) \geqslant \Delta(G) - 2c$.

Theorem (Reed, 1998)

There exists $\varepsilon > 0$ *s.t. every graph G satisfies* $\chi(G) \leqslant \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.

Theorem (Reed, 1998)

For every graph G, if $\Delta(G) \geqslant \Delta_c$ *and* $\chi(G) \geqslant \Delta(G) + 1 - c$, *then* $\omega(G) \geqslant \Delta(G) - 2c$.

Consequence: If $\Delta(G)$ is large enough and $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 2$.

Theorem (Reed, 1999)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ *and* $\Delta(G) \geqslant 10^{14}$ *then* $\omega(G) \geqslant \Delta(G)$.

Reed's conjecture

Conjecture (Reed, 1998)

Every graph G satisfies $\chi(G) \leqslant \left\lceil \frac{1}{2}(\Delta(G)+1) + \frac{1}{2}\omega(G) \right\rceil$.

Consequence: If $\chi(G) \geqslant \Delta(G) + 1 - c$ then $\omega(G) \geqslant \Delta(G) - 2c$.

Theorem (Reed, 1998)

There exists $\varepsilon > 0$ s.t. every graph G satisfies $\chi(G) \leqslant \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.

Theorem (Reed, 1998)

For every graph G, if $\Delta(G) \geqslant \Delta_c$ and $\chi(G) \geqslant \Delta(G) + 1 - c$, then $\omega(G) \geqslant \Delta(G) - 2c$.

Consequence: If $\Delta(G)$ is large enough and $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 2$.

Theorem (Reed, 1999)

For every graph G, if $\chi(G) \geqslant \Delta(G)$ and $\Delta(G) \geqslant 10^{14}$ then $\omega(G) \geqslant \Delta(G)$.

Reed's conjecture

Conjecture (Reed, 1998)

Every graph G satisfies $\chi(G) \leqslant \left\lceil \frac{1}{2}(\Delta(G)+1) + \frac{1}{2}\omega(G) \right\rceil$.

Consequence: If $\chi(G) \geqslant \Delta(G) + 1 - c$ then $\omega(G) \geqslant \Delta(G) - 2c$.

Theorem (Reed, 1998)

There exists $\varepsilon > 0$ *s.t. every graph* G *satisfies* $\chi(G) \leqslant \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.

Theorem (Reed, 1998)

For every graph G, *if* $\Delta(G) \geqslant \Delta_c$ *and* $\chi(G) \geqslant \Delta(G) + 1 - c$, *then* $\omega(G) \geqslant \Delta(G) - 2c$.

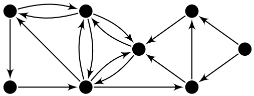
Consequence: If $\Delta(G)$ is large enough and $\chi(G) \geqslant \Delta(G)$ then $\omega(G) \geqslant \Delta(G) - 2$.

Theorem (Reed, 1999)

For every graph G, *if* $\chi(G) \geqslant \Delta(G)$ *and* $\Delta(G) \geqslant 10^{14}$ *then* $\omega(G) \geqslant \Delta(G)$.

Definition

- **Biclique number** $\overset{\leftrightarrow}{\omega}(D)$ of D: size of a largest biclique

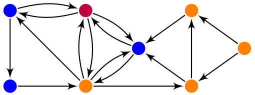


$$\overset{\leftrightarrow}{\omega}(D) = 3$$

$\overleftrightarrow{\omega}$ and $\vec{\chi}$

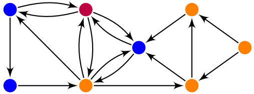
Definition

- **Biclique number** $\overleftrightarrow{\omega}(D)$ of D: size of a largest biclique 
- k**-dicolouring** of D: partition of $V(D)$ into k acyclic subdigraphs (*i.e.* no monochromatic directed cycle).



Definition

- **Biclique number** $\overset{\leftrightarrow}{\omega}(D)$ of D: size of a largest biclique 
- **k-dicolouring** of D: partition of $V(D)$ into k acyclic subdigraphs (*i.e.* no monochromatic directed cycle).
- **Dichromatic number** $\vec{\chi}(D)$ of D: minimum k s.t. D admits a k-dicolouring.

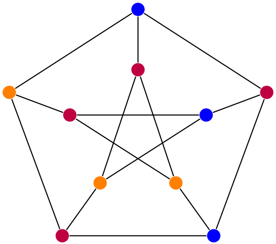


$\vec{\chi}(D) = 3$

Definition

- **Biclique number** $\overleftrightarrow{\omega}(D)$ of D: size of a largest biclique 
- k-**dicolouring** of D: partition of $V(D)$ into k acyclic subdigraphs (*i.e.* no monochromatic directed cycle).
- **Dichromatic number** $\vec{\chi}(D)$ of D: minimum k s.t. D admits a k-dicolouring.

Remark: Extensions of ω and χ.

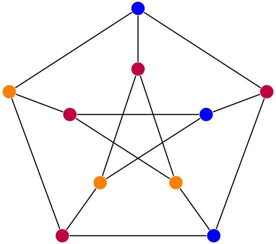


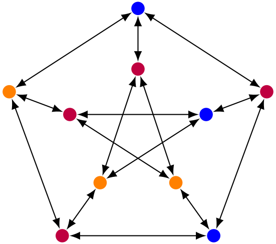
$\overleftrightarrow{\omega}$ and $\vec{\chi}$

Definition

- **Biclique number** $\overleftrightarrow{\omega}(D)$ of D: size of a largest biclique 
- k-**dicolouring** of D: partition of $V(D)$ into k acyclic subdigraphs (*i.e.* no monochromatic directed cycle).
- **Dichromatic number** $\vec{\chi}(D)$ of D: minimum k s.t. D admits a k-dicolouring.

Remark: Extensions of ω and χ.





$$\omega(G) = \overleftrightarrow{\omega}(\overleftrightarrow{G}) \quad \text{and} \quad \chi(G) = \vec{\chi}(\overleftrightarrow{G})$$

Maximum degrees of a digraph

In general: for any $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\min(a,b) \leqslant f(a,b) \leqslant \max(a,b)$,

$$\Delta_f(D) = \max_{v \in V(D)} f(d^-(v), d^+(v)).$$

Definition

- **Max-max-degree:** $\Delta_{\max}(D) = \max_v(\max(d^-(v), d^+(v)))$.
- **Max-min-degree:** $\Delta_{\min}(D) = \max_v(\min(d^-(v), d^+(v)))$.
- **Max-out-degree:** $\Delta^+(D) = \max_v(d^+(v))$.
- **Max-geometric-degree:** $\tilde{\Delta}(D) = \max_v \sqrt{d^-(v) \cdot d^+(v)}$.

Proposition

- *for every graph* G, $\Delta(G) = \Delta_f(\overleftrightarrow{G})$.
- *for every digraph* D, $\overleftrightarrow{\omega}(D) \leqslant \vec{\chi}(D) \leqslant \Delta_{\min}(D) + 1 \leqslant \Delta_{\max}(D) + 1$.

Maximum degrees of a digraph

In general: for any $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\min(a,b) \leqslant f(a,b) \leqslant \max(a,b)$,

$$\Delta_f(D) = \max_{v \in V(D)} f(d^-(v), d^+(v)).$$

Definition

- **Max-max-degree:** $\Delta_{\max}(D) = \max_v(\max(d^-(v), d^+(v)))$.
- **Max-min-degree:** $\Delta_{\min}(D) = \max_v(\min(d^-(v), d^+(v)))$.
- **Max-out-degree:** $\Delta^+(D) = \max_v(d^+(v))$.
- **Max-geometric-degree:** $\tilde{\Delta}(D) = \max_v \sqrt{d^-(v) \cdot d^+(v)}$.

Proposition

- *for every graph G,* $\Delta(G) = \Delta_f(\overleftrightarrow{G})$.
- *for every digraph D,* $\overleftrightarrow{\omega}(D) \leqslant \vec{\chi}(D) \leqslant \Delta_{\min}(D) + 1 \leqslant \Delta_{\max}(D) + 1$.

In general: for any $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\min(a,b) \leqslant f(a,b) \leqslant \max(a,b)$,

$$\Delta_f(D) = \max_{v \in V(D)} f(d^-(v), d^+(v)).$$

Definition

- **Max-max-degree:** $\Delta_{\max}(D) = \max_v(\max(d^-(v), d^+(v)))$.
- **Max-min-degree:** $\Delta_{\min}(D) = \max_v(\min(d^-(v), d^+(v)))$.
- **Max-out-degree:** $\Delta^+(D) = \max_v(d^+(v))$.
- **Max-geometric-degree:** $\tilde{\Delta}(D) = \max_v \sqrt{d^-(v) \cdot d^+(v)}$.

Proposition

- *for every graph G,* $\Delta(G) = \Delta_f(\overleftrightarrow{G})$.
- *for every digraph D,* $\overleftrightarrow{\omega}(D) \leqslant \vec{\chi}(D) \leqslant \Delta_{\min}(D) + 1 \leqslant \Delta_{\max}(D) + 1$.

Maximum degrees of a digraph

In general: for any $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\min(a,b) \leqslant f(a,b) \leqslant \max(a,b)$,

$$\Delta_f(D) = \max_{v \in V(D)} f(d^-(v), d^+(v)).$$

Definition

- **Max-max-degree:** $\Delta_{\max}(D) = \max_v(\max(d^-(v), d^+(v)))$.
- **Max-min-degree:** $\Delta_{\min}(D) = \max_v(\min(d^-(v), d^+(v)))$.
- **Max-out-degree:** $\Delta^+(D) = \max_v(d^+(v))$.
- **Max-geometric-degree:** $\tilde{\Delta}(D) = \max_v \sqrt{d^-(v) \cdot d^+(v)}$.

Proposition

- *for every graph G,* $\Delta(G) = \Delta_f(\overleftrightarrow{G})$.
- *for every digraph D,* $\overleftrightarrow{\omega}(D) \leqslant \vec{\chi}(D) \leqslant \Delta_{\min}(D) + 1 \leqslant \Delta_{\max}(D) + 1$.

Maximum degrees of a digraph

In general: for any $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\min(a,b) \leqslant f(a,b) \leqslant \max(a,b)$,

$$\Delta_f(D) = \max_{v \in V(D)} f(d^-(v), d^+(v)).$$

Definition

- **Max-max-degree:** $\Delta_{\max}(D) = \max_v(\max(d^-(v), d^+(v)))$.
- **Max-min-degree:** $\Delta_{\min}(D) = \max_v(\min(d^-(v), d^+(v)))$.
- **Max-out-degree:** $\Delta^+(D) = \max_v(d^+(v))$.
- **Max-geometric-degree:** $\tilde{\Delta}(D) = \max_v \sqrt{d^-(v) \cdot d^+(v)}$.

Proposition

- *for every graph G, $\Delta(G) = \Delta_f(\overleftrightarrow{G})$.*
- *for every digraph D, $\overleftrightarrow{\omega}(D) \leqslant \vec{\chi}(D) \leqslant \Delta_{\min}(D) + 1 \leqslant \Delta_{\max}(D) + 1$.*

Maximum degrees of a digraph

In general: for any $f : \mathbb{N}^2 \to \mathbb{N}$ such that $\min(a,b) \leqslant f(a,b) \leqslant \max(a,b)$,

$$\Delta_f(D) = \max_{v \in V(D)} f(d^-(v), d^+(v)).$$

Definition

- **Max-max-degree:** $\Delta_{\max}(D) = \max_v(\max(d^-(v), d^+(v)))$.
- **Max-min-degree:** $\Delta_{\min}(D) = \max_v(\min(d^-(v), d^+(v)))$.
- **Max-out-degree:** $\Delta^+(D) = \max_v(d^+(v))$.
- **Max-geometric-degree:** $\tilde{\Delta}(D) = \max_v \sqrt{d^-(v) \cdot d^+(v)}$.

Proposition

- *for every graph G,* $\Delta(G) = \Delta_f(\overleftrightarrow{G})$.
- *for every digraph D,* $\overleftrightarrow{\omega}(D) \leqslant \vec{\chi}(D) \leqslant \Delta_{\min}(D) + 1 \leqslant \Delta_{\max}(D) + 1$.

Maximum degrees of a digraph

In general: for any $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\min(a,b) \leqslant f(a,b) \leqslant \max(a,b)$,

$$\Delta_f(D) = \max_{v \in V(D)} f(d^-(v), d^+(v)).$$

Definition

- **Max-max-degree:** $\Delta_{\max}(D) = \max_v(\max(d^-(v), d^+(v)))$.
- **Max-min-degree:** $\Delta_{\min}(D) = \max_v(\min(d^-(v), d^+(v)))$.
- **Max-out-degree:** $\Delta^+(D) = \max_v(d^+(v))$.
- **Max-geometric-degree:** $\tilde{\Delta}(D) = \max_v \sqrt{d^-(v) \cdot d^+(v)}$.

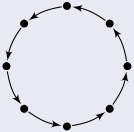
Proposition

- *for every graph G, $\Delta(G) = \Delta_f(\overleftrightarrow{G})$.*
- *for every digraph D, $\overleftrightarrow{\omega}(D) \leqslant \vec{\chi}(D) \leqslant \Delta_{\min}(D) + 1 \leqslant \Delta_{\max}(D) + 1$.*

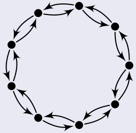
Directed Brooks' Theorems

Theorem (Mohar, 2010)

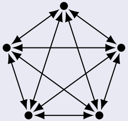
For every connected digraph D, if $\vec{\chi}(D) = \Delta_{\max}(D) + 1$ then D is a directed cycle, a symmetric odd cycle, or a complete digraph.



$\Delta_{\max} = 1,\ \vec{\chi} = 2$



$\Delta_{\max} = 2,\ \vec{\chi} = 3$



$\Delta_{\max} = n - 1,\ \vec{\chi} = n$

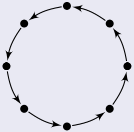
Remark: holds also for $\tilde{\Delta}$ and Δ^+, but not for $\Delta_{\min}$.

Theorem (P., 2023)

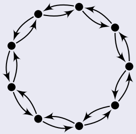
For every digraph D, if $\vec{\chi}(D) = \Delta_{\min}(D) + 1$ then $\overleftrightarrow{\omega}(D) \geqslant \frac{1}{2}(\Delta_{\min}(D) + 1)$.

Theorem (Mohar, 2010)

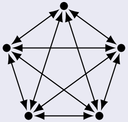
For every connected digraph D, if $\vec{\chi}(D) = \Delta_{\max}(D) + 1$ then D is a directed cycle, a symmetric odd cycle, or a complete digraph.



$\Delta_{\max} = 1,\ \vec{\chi} = 2$



$\Delta_{\max} = 2,\ \vec{\chi} = 3$



$\Delta_{\max} = n - 1,\ \vec{\chi} = n$

Remark: holds also for $\tilde{\Delta}$ and Δ^+, but not for $\Delta_{\min}$.

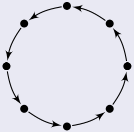
Theorem (P., 2023)

For every digraph D, if $\vec{\chi}(D) = \Delta_{\min}(D) + 1$ then $\overleftrightarrow{\omega}(D) \geqslant \frac{1}{2}(\Delta_{\min}(D) + 1)$.

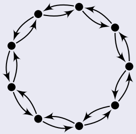
Directed Brooks' Theorems

Theorem (Mohar, 2010)

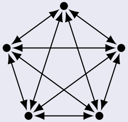
For every connected digraph D, if $\vec{\chi}(D) = \Delta_{\max}(D) + 1$ then D is a directed cycle, a symmetric odd cycle, or a complete digraph.



$\Delta_{\max} = 1,\ \vec{\chi} = 2$



$\Delta_{\max} = 2,\ \vec{\chi} = 3$



$\Delta_{\max} = n - 1,\ \vec{\chi} = n$

Remark: holds also for $\tilde{\Delta}$ and Δ^+, but not for $\Delta_{\min}$.

Theorem (P., 2023)

For every digraph D, if $\vec{\chi}(D) = \Delta_{\min}(D) + 1$ then $\overleftrightarrow{\omega}(D) \geqslant \frac{1}{2}(\Delta_{\min}(D) + 1)$.

An analogue of Reed's conjecture for digraphs

Conjecture (Kawarabayashi and P., 2025)

Every digraph D satisfies $\vec{\chi}(D) \leqslant \lceil \frac{1}{2}(\tilde{\Delta}(D) + 1) + \frac{1}{2}\overleftrightarrow{\omega}(D) \rceil$.

Theorem (Kawarabayashi and P., 2025)

There exists $\varepsilon > 0$ s.t. every digraph D satisfies $\vec{\chi}(D) \leqslant \lceil (1 - \varepsilon)(\tilde{\Delta}(D) + 1) + \varepsilon \overleftrightarrow{\omega}(D) \rceil$.

Corollary

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D) - 2/\varepsilon$.

But $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ **does not imply** $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$, even for large values of $\tilde{\Delta}$, because of

$$H_\Delta = \vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}.$$

$\overleftrightarrow{\omega} = n - 2$

$\tilde{\Delta} = n - 1$

$\vec{\chi} = n - 1$

An analogue of Reed's conjecture for digraphs

Conjecture (Kawarabayashi and P., 2025)

Every digraph D satisfies $\vec{\chi}(D) \leqslant \lceil \frac{1}{2}(\tilde{\Delta}(D) + 1) + \frac{1}{2}\overleftrightarrow{\omega}(D) \rceil$.

Theorem (Kawarabayashi and P., 2025)

There exists $\varepsilon > 0$ *s.t. every digraph D satisfies* $\vec{\chi}(D) \leqslant \lceil (1-\varepsilon)(\tilde{\Delta}(D) + 1) + \varepsilon \overleftrightarrow{\omega}(D) \rceil$.

Corollary

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D) - 2/\varepsilon$.

But $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ **does not imply** $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$, even for large values of $\tilde{\Delta}$, because of

$$H_\Delta = \vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}.$$

$\overleftrightarrow{\omega} = n - 2$

$\tilde{\Delta} = n - 1$

$\vec{\chi} = n - 1$

An analogue of Reed's conjecture for digraphs

Conjecture (Kawarabayashi and P., 2025)

Every digraph D satisfies $\vec{\chi}(D) \leqslant \lceil \frac{1}{2}(\tilde{\Delta}(D) + 1) + \frac{1}{2}\overleftrightarrow{\omega}(D) \rceil$.

Theorem (Kawarabayashi and P., 2025)

There exists $\varepsilon > 0$ *s.t. every digraph* D *satisfies* $\vec{\chi}(D) \leqslant \lceil (1-\varepsilon)(\tilde{\Delta}(D) + 1) + \varepsilon \overleftrightarrow{\omega}(D) \rceil$.

Corollary

For every digraph D*, if* $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D) - 2/\varepsilon$.

But $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ **does not imply** $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$, even for large values of $\tilde{\Delta}$, because of

$$H_\Delta = \vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}.$$

$\overleftrightarrow{\omega} = n - 2$

$\tilde{\Delta} = n - 1$

$\vec{\chi} = n - 1$

An analogue of Reed's conjecture for digraphs

Conjecture (Kawarabayashi and P., 2025)

Every digraph D satisfies $\vec{\chi}(D) \leqslant \lceil \frac{1}{2}(\tilde{\Delta}(D)+1) + \frac{1}{2}\overleftrightarrow{\omega}(D) \rceil$.

Theorem (Kawarabayashi and P., 2025)

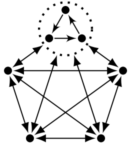
There exists $\varepsilon > 0$ s.t. every digraph D satisfies $\vec{\chi}(D) \leqslant \lceil (1-\varepsilon)(\tilde{\Delta}(D)+1) + \varepsilon\overleftrightarrow{\omega}(D) \rceil$.

Corollary

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D) - 2/\varepsilon$.

But $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ **does not imply** $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$, even for large values of $\tilde{\Delta}$, because of

$$H_\Delta = \vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}.$$



$\overleftrightarrow{\omega} = n-2$

$\tilde{\Delta} = n-1$

$\vec{\chi} = n-1$

Directed analogues of Borodin–Kostochka's conjecture

Conjecture (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- *If* $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *and* $\tilde{\Delta}(D) \geqslant 9$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ *or* $H_{\tilde{\Delta}(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta^+(D)$ *and* $\Delta^+(D) \geqslant 9$ *then* $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ *or* $H_{\Delta^+(D)} \subseteq D$.

Theorem (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- *If* $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *and* $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ *or* $H_{\tilde{\Delta}(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta^+(D)$ *and* $\Delta^+(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ *or* $H_{\Delta^+(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta_{\min}(D)$ *and* $\Delta_{\min}(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \frac{1}{2}(\Delta_{\min}(D) - 1)$.

Conjecture (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- *If* $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *and* $\tilde{\Delta}(D) \geqslant 9$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ *or* $H_{\tilde{\Delta}(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta^+(D)$ *and* $\Delta^+(D) \geqslant 9$ *then* $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ *or* $H_{\Delta^+(D)} \subseteq D$.

Theorem (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- *If* $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *and* $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ *or* $H_{\tilde{\Delta}(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta^+(D)$ *and* $\Delta^+(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ *or* $H_{\Delta^+(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta_{\min}(D)$ *and* $\Delta_{\min}(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \frac{1}{2}(\Delta_{\min}(D) - 1)$.

Directed analogues of Borodin–Kostochka's conjecture

Conjecture (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- *If* $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *and* $\tilde{\Delta}(D) \geqslant 9$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ *or* $H_{\tilde{\Delta}(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta^+(D)$ *and* $\Delta^+(D) \geqslant 9$ *then* $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ *or* $H_{\Delta^+(D)} \subseteq D$.

Theorem (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- *If* $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ *and* $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ *or* $H_{\tilde{\Delta}(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta^+(D)$ *and* $\Delta^+(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ *or* $H_{\Delta^+(D)} \subseteq D$.
- *If* $\vec{\chi}(D) \geqslant \Delta_{\min}(D)$ *and* $\Delta_{\min}(D) \geqslant 10^{10^{10}}$ *then* $\overleftrightarrow{\omega}(D) \geqslant \frac{1}{2}(\Delta_{\min}(D) - 1)$.

Conjecture (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- If $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ and $\tilde{\Delta}(D) \geqslant 9$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ or $H_{\tilde{\Delta}(D)} \subseteq D$.
- If $\vec{\chi}(D) \geqslant \Delta^+(D)$ and $\Delta^+(D) \geqslant 9$ then $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ or $H_{\Delta^+(D)} \subseteq D$.

Theorem (Harutyunyan, Kawarabayashi, P., Puig i Surroca)

The following hold for every digraph D.

- If $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ and $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ or $H_{\tilde{\Delta}(D)} \subseteq D$.
- If $\vec{\chi}(D) \geqslant \Delta^+(D)$ and $\Delta^+(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \Delta^+(D)$ or $H_{\Delta^+(D)} \subseteq D$.
- If $\vec{\chi}(D) \geqslant \Delta_{\min}(D)$ and $\Delta_{\min}(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \frac{1}{2}(\Delta_{\min}(D) - 1)$.

Overview of the proof

1. Proof of a **Dense Decomposition Lemma** for digraphs of bounded maximum degree.
2. Proof of the result for $\tilde{\Delta}$.
   i. Apply the **DDL** to a minimum counterexample to restrain its possible structure.
   ii. Show that a (pseudo-) **random colouring** uses at most $\lceil \tilde{\Delta} - 1 \rceil$ **colours** with positive probability.
3. For Δ^+, show that a (well-chosen) minimum counter-example satisfies $\Delta^+ \geqslant \tilde{\Delta}$.
4. Derive the result for $\Delta_{\min}$ from the one for Δ^+.

Overview of the proof

1. Proof of a **Dense Decomposition Lemma** for digraphs of bounded maximum degree.
2. Proof of the result for $\tilde{\Delta}$.
   i. Apply the **DDL** to a minimum counterexample to restrain its possible structure.
   ii. Show that a (pseudo-) **random colouring** uses at most $\lceil \tilde{\Delta} - 1 \rceil$ **colours** with positive probability.
3. For Δ^+, show that a (well-chosen) minimum counter-example satisfies $\Delta^+ \geqslant \tilde{\Delta}$.
4. Derive the result for $\Delta_{\min}$ from the one for Δ^+.

Overview of the proof

1. Proof of a **Dense Decomposition Lemma** for digraphs of bounded maximum degree.
2. Proof of the result for $\tilde{\Delta}$.
   i. Apply the **DDL** to a minimum counterexample to restrain its possible structure.
   ii. Show that a (pseudo-) **random colouring** uses at most $\lceil \tilde{\Delta} - 1 \rceil$ **colours** with positive probability.
3. For Δ^+, show that a (well-chosen) minimum counter-example satisfies $\Delta^+ \geqslant \tilde{\Delta}$.
4. Derive the result for $\Delta_{\min}$ from the one for Δ^+.

Overview of the proof

1. Proof of a **Dense Decomposition Lemma** for digraphs of bounded maximum degree.
2. Proof of the result for $\tilde{\Delta}$.
   i. Apply the **DDL** to a minimum counterexample to restrain its possible structure.
   ii. Show that a (pseudo-) **random colouring** uses at most $\lceil \tilde{\Delta} - 1 \rceil$ **colours** with positive probability.
3. For Δ^+, show that a (well-chosen) minimum counter-example satisfies $\Delta^+ \geqslant \tilde{\Delta}$.
4. Derive the result for $\Delta_{\min}$ from the one for Δ^+.

Overview of the proof

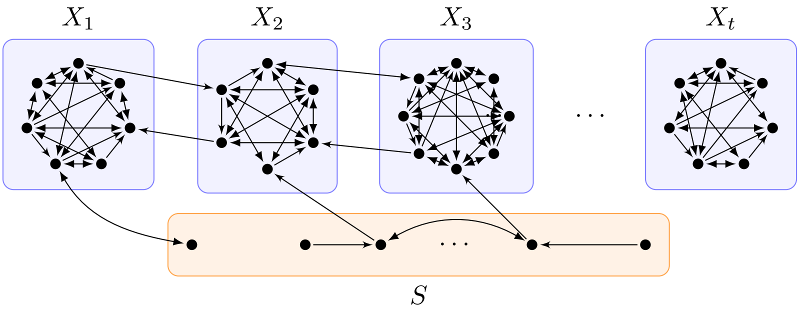
1. Proof of a **Dense Decomposition Lemma** for digraphs of bounded maximum degree.
2. Proof of the result for $\tilde{\Delta}$.
   i. Apply the **DDL** to a minimum counterexample to restrain its possible structure.
   ii. Show that a (pseudo-) **random colouring** uses at most $\lceil \tilde{\Delta} - 1 \rceil$ **colours** with positive probability.
3. For Δ^+, show that a (well-chosen) minimum counter-example satisfies $\Delta^+ \geqslant \tilde{\Delta}$.
4. Derive the result for $\Delta_{\min}$ from the one for Δ^+.

A Dense Decomposition Lemma for digraphs

Lemma (Dense Decomposition)

For every $0 < \varepsilon < \frac{1}{2}$ *and* $\omega(1) \leqslant d \leqslant o(\Delta_{\max})$, *there exists* Δ_0 *s.t. every digraph* D *with* $\Delta_{\max}(D) = \Delta_{\max} \geqslant \Delta_0$ *admits a vertex-partition* $V(D) = (X_1, \dots, X_t, S)$ *s.t.:*

1. *for every* $i \in [t]$, $\Delta_{\max} - \frac{3}{\varepsilon}d < |X_i| < \Delta_{\max} + 1 + 4d$;
2. *for every* $i \in [t]$ *and* $u \in V(D)$, $u \in X_i$ *iff* $|N^+(u) \cap X_i| \geqslant (1-\varepsilon)\Delta_{\max}$; *and*
3. *vertices in* S *are* d*-sparse*.

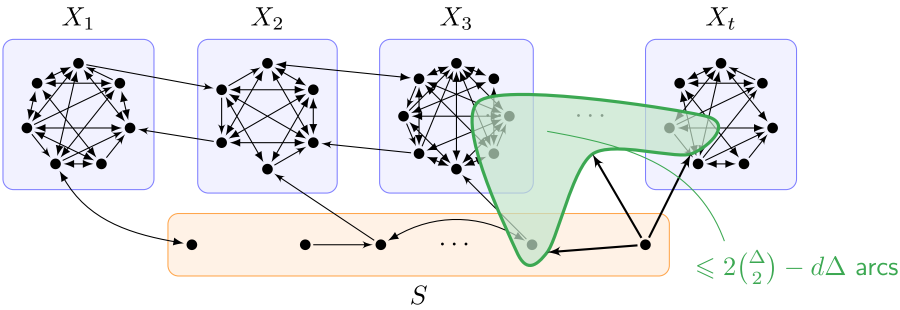


A Dense Decomposition Lemma for digraphs

Lemma (Dense Decomposition)

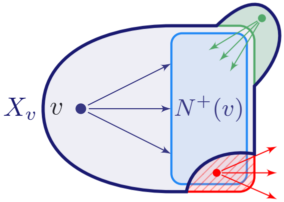
For every $0 < \varepsilon < \frac{1}{2}$ *and* $\omega(1) \leqslant d \leqslant o(\Delta_{\max})$, *there exists* Δ_0 *s.t. every digraph* D *with* $\Delta_{\max}(D) = \Delta_{\max} \geqslant \Delta_0$ *admits a vertex-partition* $V(D) = (X_1, \dots, X_t, S)$ *s.t.:*

1. *for every* $i \in [t]$, $\Delta_{\max} - \frac{3}{\varepsilon}d < |X_i| < \Delta_{\max} + 1 + 4d$;
2. *for every* $i \in [t]$ *and* $u \in V(D)$, $u \in X_i$ *iff* $|N^+(u) \cap X_i| \geqslant (1-\varepsilon)\Delta_{\max}$; *and*
3. *vertices in* S *are* d*-sparse.*



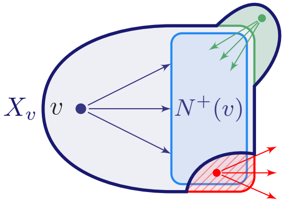
Dense Decomposition Lemma – Sketch of proof

- For every d-**dense** vertex $v \in \mathscr{D}$, construct X_v as follows.
  1. Start with $X_v = N^+[v]$.
  2. Remove from X_v all vertices u with $|N^+(u) \cap X_v| < (1-\varepsilon)\Delta$.
  3. Add to X_v all vertices u with $|N^+(u) \cap X_v| \geqslant (1-\varepsilon)\Delta$.



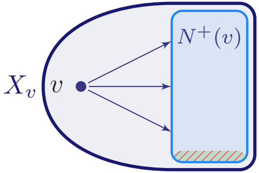
Dense Decomposition Lemma – Sketch of proof

- For every d-**dense** vertex $v \in \mathscr{D}$, construct X_v as follows.
  1. Start with $X_v = N^+[v]$.
  2. Remove from X_v all vertices u with $|N^+(u) \cap X_v| < (1-\varepsilon)\Delta$.
  3. Add to X_v all vertices u with $|N^+(u) \cap X_v| \geqslant (1-\varepsilon)\Delta$.
- At most $O(d/\varepsilon)$ vertices are removed / added at steps 2 and 3.



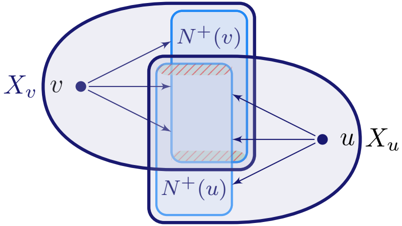
Dense Decomposition Lemma – Sketch of proof

- For every d-**dense** vertex $v \in \mathscr{D}$, construct X_v as follows.
  1. Start with $X_v = N^+[v]$.
  2. Remove from X_v all vertices u with $|N^+(u) \cap X_v| < (1-\varepsilon)\Delta$.
  3. Add to X_v all vertices u with $|N^+(u) \cap X_v| \geqslant (1-\varepsilon)\Delta$.
- At most $O(d/\varepsilon)$ vertices are removed / added at steps 2 and 3.
- Vertices outside $\bigcup_{v \in \mathscr{D}} X_v$ are sparse.



Dense Decomposition Lemma – Sketch of proof

- For every **d-dense** vertex $v \in \mathscr{D}$, construct X_v as follows.
  1. Start with $X_v = N^+[v]$.
  2. Remove from X_v all vertices u with $|N^+(u) \cap X_v| < (1-\varepsilon)\Delta$.
  3. Add to X_v all vertices u with $|N^+(u) \cap X_v| \geqslant (1-\varepsilon)\Delta$.
- At most $O(d/\varepsilon)$ vertices are removed / added at steps 2 and 3.
- Vertices outside $\bigcup_{v\in\mathscr{D}} X_v$ are sparse.
- For every $u, v \in \mathscr{D}$, if $X_u \cap X_v \neq \varnothing$ then $u \in X_v$ and $v \in X_u$.



Proof for $\tilde{\Delta}$: applying the DDL – an illustration

Theorem

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ and $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ or $H_{\tilde{\Delta}(D)} \subseteq D$.

Claim: If D is a minimum counterexample, then $\Delta_{\max}(D) \leqslant \tilde{\Delta}(D) + 1$.
(Because $(\tilde{\Delta} - 1)(\tilde{\Delta} + 2) > \tilde{\Delta}^2$.)

Claim: For every $i \in [t]$, $\bar{D}[X_i]$ has no **matching** of size $2\varepsilon\tilde{\Delta}$.

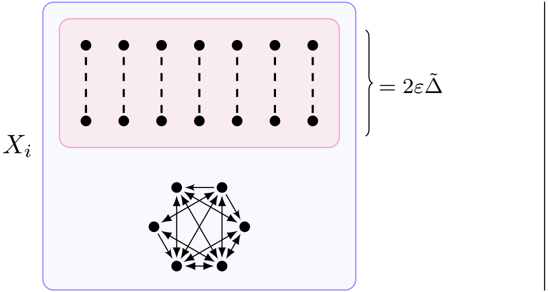
Proof for $\tilde{\Delta}$: applying the DDL – an illustration

Theorem

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ and $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ or $H_{\tilde{\Delta}(D)} \subseteq D$.

Claim: If D is a minimum counterexample, then $\Delta_{\max}(D) \leqslant \tilde{\Delta}(D) + 1$.
(Because $(\tilde{\Delta} - 1)(\tilde{\Delta} + 2) > \tilde{\Delta}^2$.)

Claim: For every $i \in [t]$, $\bar{D}[X_i]$ has no **matching** of size $2\varepsilon\tilde{\Delta}$.



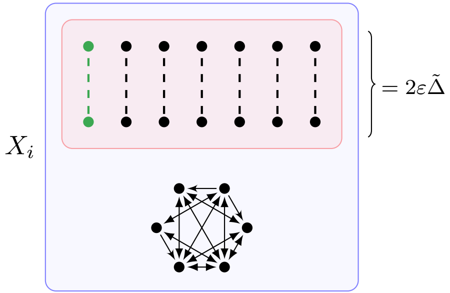
Proof for $\tilde{\Delta}$: applying the DDL – an illustration

Theorem

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ and $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ or $H_{\tilde{\Delta}(D)} \subseteq D$.

Claim: If D is a minimum counterexample, then $\Delta_{\max}(D) \leqslant \tilde{\Delta}(D) + 1$.
(Because $(\tilde{\Delta} - 1)(\tilde{\Delta} + 2) > \tilde{\Delta}^2$.)

Claim: For every $i \in [t]$, $\bar{D}[X_i]$ has no **matching** of size $2\varepsilon\tilde{\Delta}$.



Number of forbidden colours:

- $\leqslant 2\varepsilon\tilde{\Delta}$

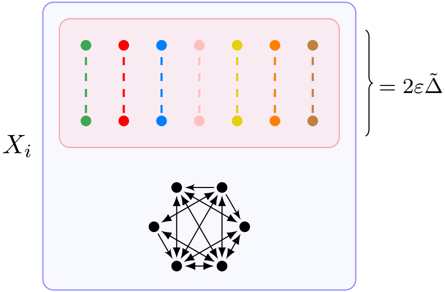
Proof for $\tilde{\Delta}$: applying the DDL – an illustration

Theorem

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ and $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ or $H_{\tilde{\Delta}(D)} \subseteq D$.

Claim: If D is a minimum counterexample, then $\Delta_{\max}(D) \leqslant \tilde{\Delta}(D) + 1$.
(Because $(\tilde{\Delta} - 1)(\tilde{\Delta} + 2) > \tilde{\Delta}^2$.)

Claim: For every $i \in [t]$, $\bar{D}[X_i]$ has no **matching** of size $2\varepsilon\tilde{\Delta}$.



Number of forbidden colours:

- $\leqslant 2\varepsilon\tilde{\Delta}$
- $\leqslant 4\varepsilon\tilde{\Delta}$

Proof for $\tilde{\Delta}$: applying the DDL – an illustration

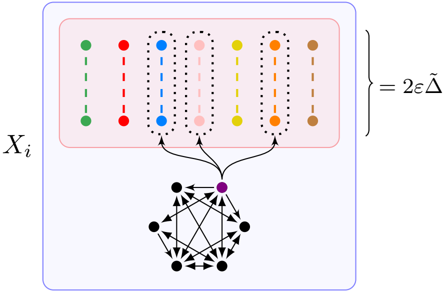
Theorem

For every digraph D, if $\vec{\chi}(D) \geqslant \tilde{\Delta}(D)$ and $\tilde{\Delta}(D) \geqslant 10^{10^{10}}$ then $\overleftrightarrow{\omega}(D) \geqslant \tilde{\Delta}(D)$ or $H_{\tilde{\Delta}(D)} \subseteq D$.

Claim: If D is a minimum counterexample, then $\Delta_{\max}(D) \leqslant \tilde{\Delta}(D) + 1$.

(Because $(\tilde{\Delta} - 1)(\tilde{\Delta} + 2) > \tilde{\Delta}^2$.)

Claim: For every $i \in [t]$, $\bar{D}[X_i]$ has no **matching** of size $2\varepsilon\tilde{\Delta}$.



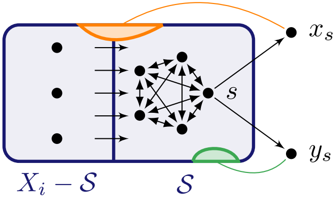
Number of forbidden colours:

- $\leqslant 2\varepsilon\tilde{\Delta}$
- $\leqslant 4\varepsilon\tilde{\Delta}$
- $\leqslant (\tilde{\Delta} + 1) - 3 = \tilde{\Delta} - 2$

Proof for $\tilde{\Delta}$: applying the DDL – what we actually obtain

Lemma

If D is a minimum counterexample with dense decomposition $(X_1, \ldots, X_t, S)$, then each X_i contains at least $\frac{1}{3}|X_i|$ **saviours**.

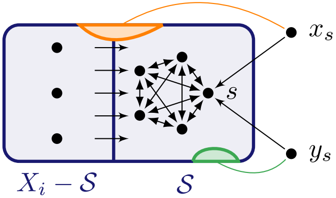


- s is dominated by X_i,
- $d^+(s) = \lceil \tilde{\Delta} \rceil$,
- x_s, y_s have at most $\log^4 \tilde{\Delta}$ neighbours in X_i.

Proof for $\tilde{\Delta}$: applying the DDL – what we actually obtain

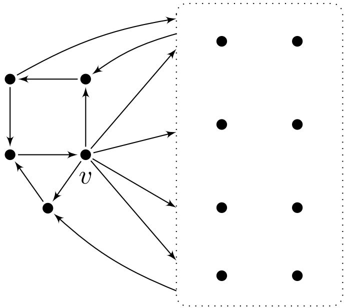
Lemma

If D is a minimum counterexample with dense decomposition $(X_1, \dots, X_t, S)$, then each X_i contains at least $\frac{1}{3}|X_i|$ **saviours**.



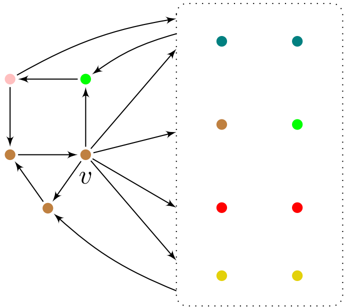
- s is dominated by X_i,
- $d^-(s) = \lceil \tilde{\Delta} \rceil$,
- x_s, y_s have at most $\log^4 \tilde{\Delta}$ neighbours in X_i.

1. Colour **uniformly at random** with $\{1, \lceil \tilde{\Delta} - 1 \rceil\}$.



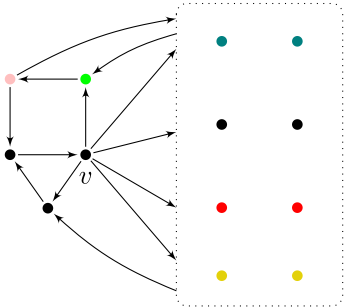
Proof for $\tilde{\Delta}$: the pseudo-random colouring process

1. Colour **uniformly at random** with $\{1, \lceil \tilde{\Delta} - 1 \rceil\}$.



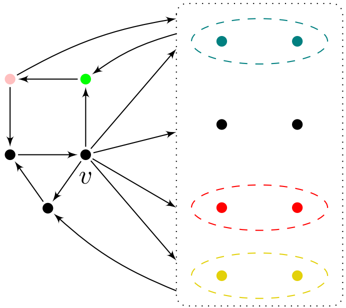
Proof for $\tilde{\Delta}$: the pseudo-random colouring process

1. Colour **uniformly at random** with $\{1, \lceil\tilde{\Delta} - 1\rceil\}$.
2. **Uncolour** every vertex with both an in-neighbour and an out-neighbour of the same colour.



Proof for $\tilde{\Delta}$: the pseudo-random colouring process

1. Colour **uniformly at random** with $\{1, \lceil \tilde{\Delta} - 1 \rceil\}$.
2. **Uncolour** every vertex with both an in-neighbour and an out-neighbour of the same colour.



Claim: A sparse vertex v has at least **three repeated colours** in its out-neighbourhood with probability at least $1 - e^{\log^2 \tilde{\Delta}}$.

- The expected number of repeated colours is large; $\Rightarrow$ conclude with Talagrand's Inequality.
- In particular, w.h.p., the colouring can be extended to v.

Proof for $\tilde{\Delta}$: the pseudo-random colouring process

Claim: For every $i \in [t]$, **at least 3 saviours** of X_i are actually **saving** X_i with probability at least $1 - e^{-\tilde{\Delta}/\log^4 \tilde{\Delta}}$.

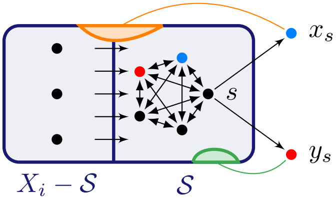
- s is uncoloured,
- x_s and y_s remain coloured, and
- the colour of x_s and y_s appear in $N^+(s) \cap X_i$.

- The expected number of actually saving saviours is large;
  $\Rightarrow$ conclude with Azuma's Inequality.
- In particular, w.h.p., the colouring can be extended to X_i.

- Each **bad event** occurs with probability at most $e^{-\log^2 \tilde{\Delta}} = p$ and is mutually independent from all others, except $\gamma = O(\tilde{\Delta}^5)$ of them.
- Since $e \cdot p \cdot (\gamma + 1) \leqslant 1$, conclude with **Lovász Local Lemma**.

Proof for $\tilde{\Delta}$: the pseudo-random colouring process

Claim: For every $i \in [t]$, **at least 3 saviours** of X_i are actually **saving X_i** with probability at least $1 - e^{-\tilde{\Delta}/\log^4 \tilde{\Delta}}$.

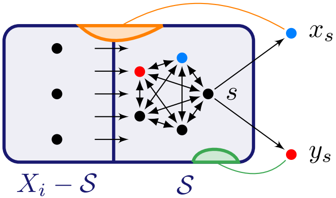


- s is uncoloured,
- x_s and y_s remain coloured, and
- the colour of x_s and y_s appear in $N^+(s) \cap X_i$.

- The expected number of actually saving saviours is large;
  $\Rightarrow$ conclude with Azuma's Inequality.
- In particular, w.h.p., the colouring can be extended to X_i.

- Each **bad event** occurs with probability at most $e^{-\log^2 \tilde{\Delta}} = p$ and is mutually independent from all others, except $\gamma = O(\tilde{\Delta}^5)$ of them.
- Since $e \cdot p \cdot (\gamma + 1) \leqslant 1$, conclude with **Lovász Local Lemma**.

Claim: For every $i \in [t]$, **at least 3 saviours** of X_i are actually **saving X_i** with probability at least $1 - e^{-\tilde{\Delta}/\log^4 \tilde{\Delta}}$.



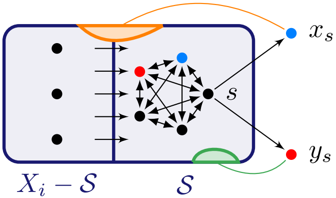
- s is uncoloured,
- x_s and y_s remain coloured, and
- the colour of x_s and y_s appear in $N^+(s) \cap X_i$.

- The expected number of actually saving saviours is large;
  $\Rightarrow$ conclude with Azuma's Inequality.
- In particular, w.h.p., the colouring can be extended to X_i.

- Each **bad event** occurs with probability at most $e^{-\log^2 \tilde{\Delta}} = p$ and is mutually independent from all others, except $\gamma = O(\tilde{\Delta}^5)$ of them.
- Since $e \cdot p \cdot (\gamma + 1) \leqslant 1$, conclude with **Lovász Local Lemma**.

Proof for $\tilde{\Delta}$: the pseudo-random colouring process

Claim: For every $i \in [t]$, **at least 3 saviours** of X_i are actually **saving X_i** with probability at least $1 - e^{-\tilde{\Delta}/\log^4 \tilde{\Delta}}$.



- s is uncoloured,
- x_s and y_s remain coloured, and
- the colour of x_s and y_s appear in $N^+(s) \cap X_i$.

- The expected number of actually saving saviours is large;
  $\Rightarrow$ conclude with Azuma's Inequality.
- In particular, w.h.p., the colouring can be extended to X_i.

- Each **bad event** occurs with probability at most $e^{-\log^2 \tilde{\Delta}} = p$ and is mutually independent from all others, except $\gamma = O(\tilde{\Delta}^5)$ of them.
- Since $e \cdot p \cdot (\gamma + 1) \leqslant 1$, conclude with **Lovász Local Lemma**.

Open problems

Problem

*For every $\Delta \geqslant \Delta_k$, the set of $(\Delta+1-k)$-***critical digraphs*** with maximum degree Δ is* **finite**.

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^{+}\}$. This might hold whenever $(k+1)(k+2) \leqslant \Delta$.

Conjecture (Erdős and Neumann-Lara, 1979)

Oriented graphs *D have dichromatic number at most $O\left(\frac{\Delta(D)}{\log \Delta(D)}\right)$.*

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^{+}, \Delta_{\min}\}$.

Problem (Kawarabayashi and P., 2025)

Oriented graphs *D have dichromatic number at most $(1-\varepsilon)\Delta^{+}(D) + O(1)$.*

Remark: Oriented graphs D satisfy $\vec{\chi}(D) \leqslant \frac{2}{3}\Delta_{\max}(D) + O(1)$ and $\vec{\chi}(D) \leqslant \frac{\sqrt{2}}{2}\tilde{\Delta}(D) + O(1)$.

Thank You!

Open problems

Problem

For every $\Delta \geqslant \Delta_k$, the set of $(\Delta+1-k)$**-critical digraphs** *with maximum degree Δ is* **finite**.

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^+\}$. This might hold whenever $(k+1)(k+2) \leqslant \Delta$.

Conjecture (Erdős and Neumann-Lara, 1979)

Oriented graphs D *have dichromatic number at most* $O\left(\frac{\Delta(D)}{\log \Delta(D)}\right)$.

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^+, \Delta_{\min}\}$.

Problem (Kawarabayashi and P., 2025)

Oriented graphs D *have dichromatic number at most* $(1-\varepsilon)\Delta^+(D) + O(1)$.

Remark: Oriented graphs D satisfy $\vec{\chi}(D) \leqslant \frac{2}{3}\Delta_{\max}(D) + O(1)$ and $\vec{\chi}(D) \leqslant \frac{\sqrt{2}}{2}\tilde{\Delta}(D) + O(1)$.

Thank You!

Open problems

Problem

For every $\Delta \geqslant \Delta_k$, the set of $(\Delta + 1 - k)$-**critical digraphs** *with maximum degree Δ is* **finite**.

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^+\}$. This might hold whenever $(k+1)(k+2) \leqslant \Delta$.

Conjecture (Erdős and Neumann-Lara, 1979)

Oriented graphs *D have dichromatic number at most* $O\left(\frac{\Delta(D)}{\log \Delta(D)}\right)$.

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^+, \Delta_{\min}\}$.

Problem (Kawarabayashi and P., 2025)

Oriented graphs *D have dichromatic number at most* $(1 - \varepsilon)\Delta^+(D) + O(1)$.

Remark: Oriented graphs D satisfy $\vec{\chi}(D) \leqslant \frac{2}{3}\Delta_{\max}(D) + O(1)$ and $\vec{\chi}(D) \leqslant \frac{\sqrt{2}}{2}\tilde{\Delta}(D) + O(1)$.

Thank You!

Open problems

Problem

For every $\Delta \geqslant \Delta_k$, the set of $(\Delta+1-k)$-**critical digraphs** *with maximum degree Δ is* **finite**.

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^+\}$. This might hold whenever $(k+1)(k+2) \leqslant \Delta$.

Conjecture (Erdős and Neumann-Lara, 1979)

Oriented graphs *D have dichromatic number at most* $O\left(\frac{\Delta(D)}{\log \Delta(D)}\right)$.

Remark: open for any $\Delta \in \{\Delta_{\max}, \tilde{\Delta}, \Delta^+, \Delta_{\min}\}$.

Problem (Kawarabayashi and P., 2025)

Oriented graphs *D have dichromatic number at most* $(1-\varepsilon)\Delta^+(D) + O(1)$.

Remark: Oriented graphs D satisfy $\vec{\chi}(D) \leqslant \frac{2}{3}\Delta_{\max}(D) + O(1)$ and $\vec{\chi}(D) \leqslant \frac{\sqrt{2}}{2}\tilde{\Delta}(D) + O(1)$.

Thank You!