# Edge-colouring and orientations: applications to degree- and $\chi$-boundedness

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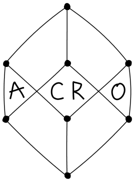

# Ramsey Theorem

**Ramsey Number** $R(s,t)$ : min. integer $n$ such that all (blue/red)-edge-colourings of $K_n$ contains $K_s$ **in red** or $K_t$ **in blue**.





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**Question:** What can we say about the **monochromatic induced substructures** in general edge-coloured graphs ?

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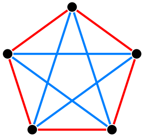

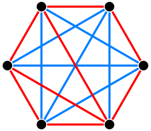

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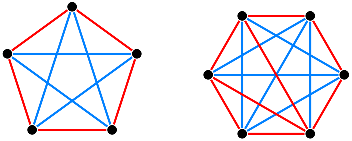

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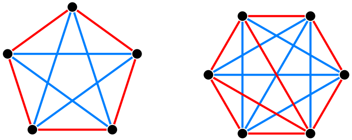

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**Theorem (Kővári, Sós, and Turán, 1954)**

*For every graph $G$ of order $n$, if $K_{s,s} \not\subseteq G$ then $G$ has at most $f(s) \cdot n^{2-\frac{1}{s}}$ edges.*

**Corollary**

*For every $\varepsilon > 0$, if $G$ is a 2-edge-coloured graph of order $n \geqslant f(\varepsilon, s, t)$ with at least $\varepsilon \cdot n^2$ edges, then $G$ contains a monochromatic induced copy of $K_{s,s}$ or $K_t$.*

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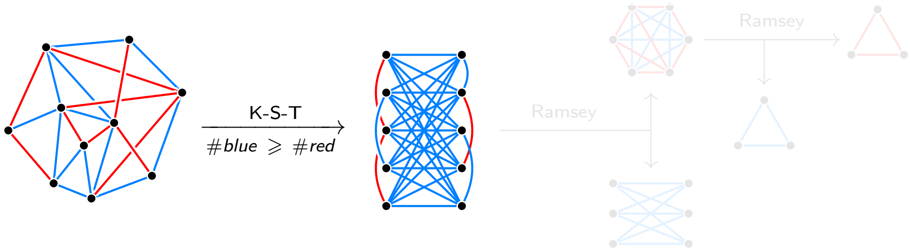

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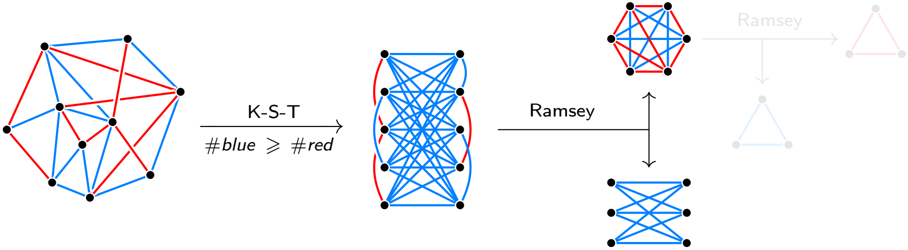

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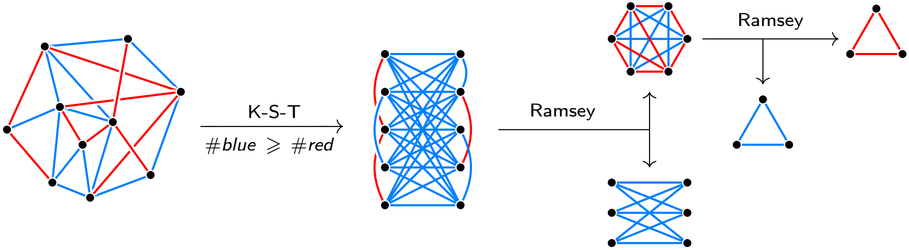

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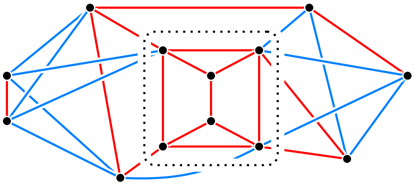
**Theorem (Char, Kawarabayashi, P-A, 2025)**

*If $G$ is a 2-edge-coloured graph with $\delta(G) \geqslant f(d)$ then $G$ contains a* **monochromatic induced subgraph** *$H$ with $\delta(H) \geqslant d$.*



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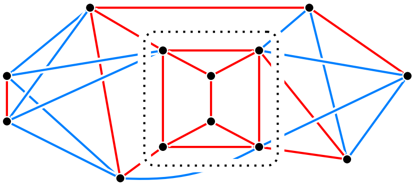

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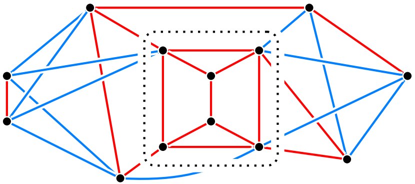

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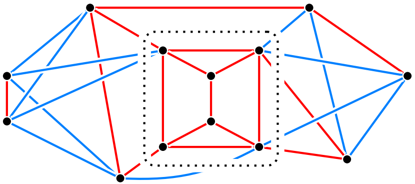

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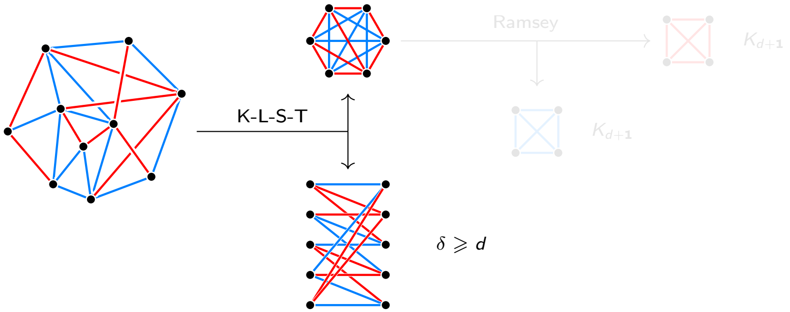

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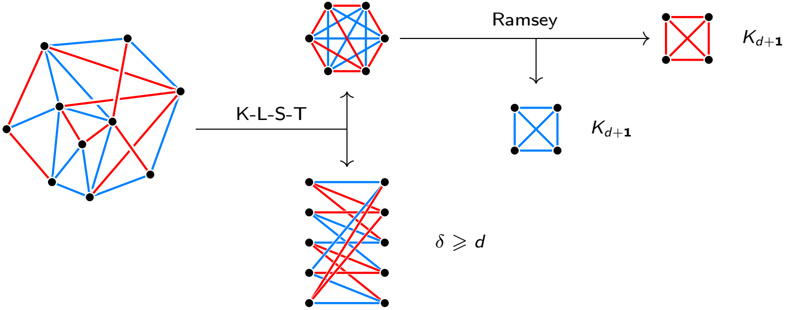

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**Lemma (Kühn and Osthus, 2004)**

*Every bipartite graph* $G = (A \cup B, E)$ *with average degree* $Ad(G) = \Gamma > 4d \geqslant 32$ *contains an* **induced bipartite subgraph** $G' = (A' \cup B', E')$ *such that:*

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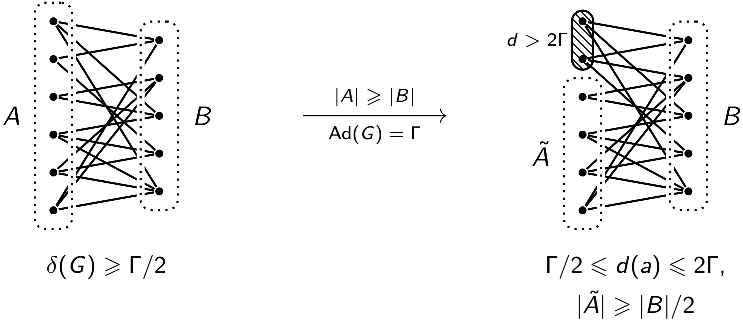

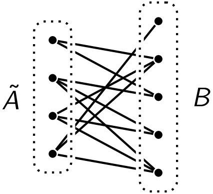


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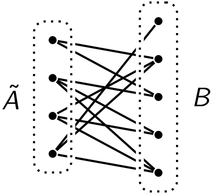


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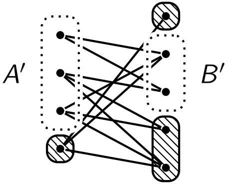


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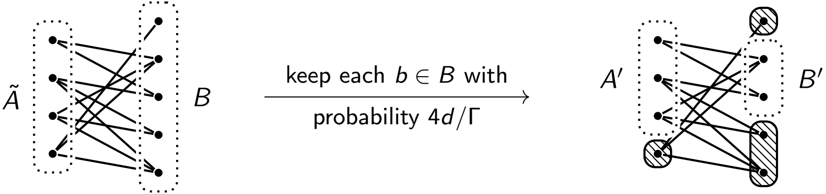


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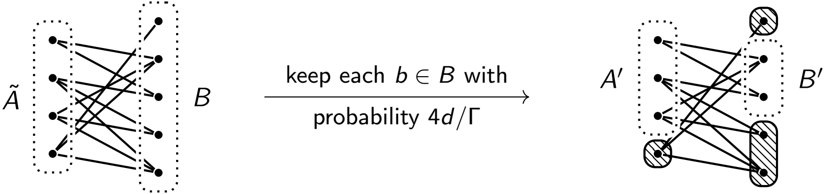


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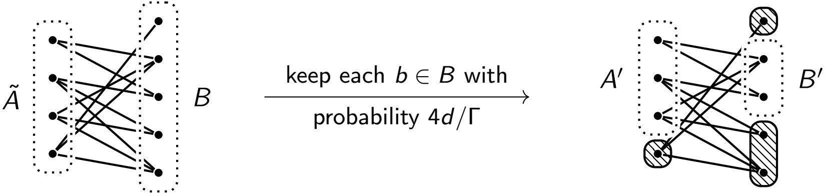


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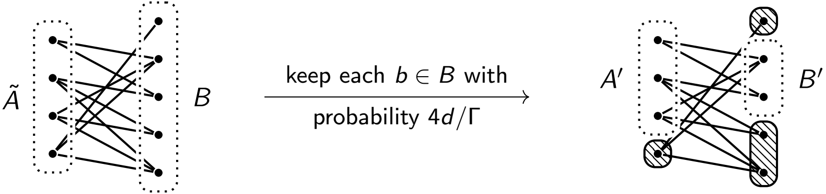


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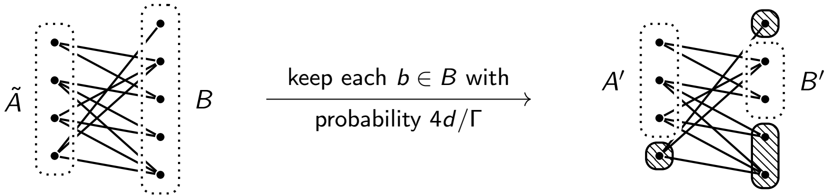


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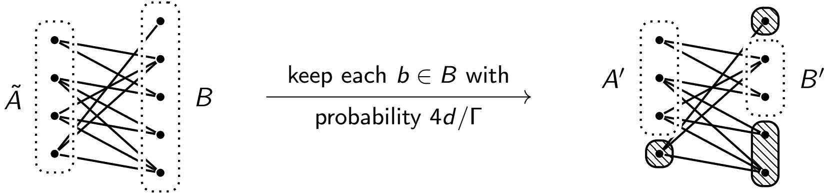


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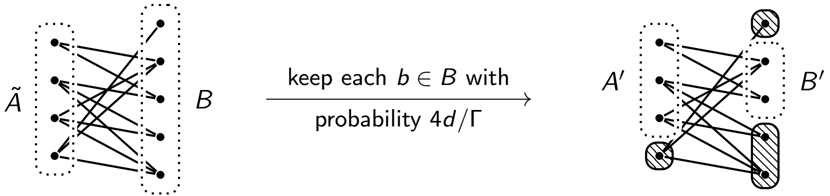


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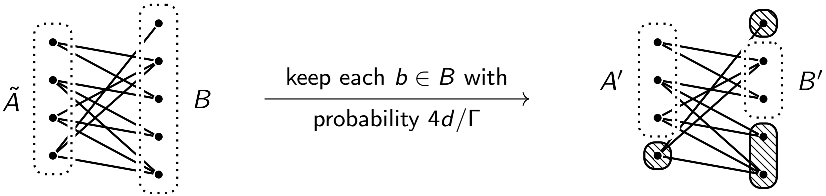


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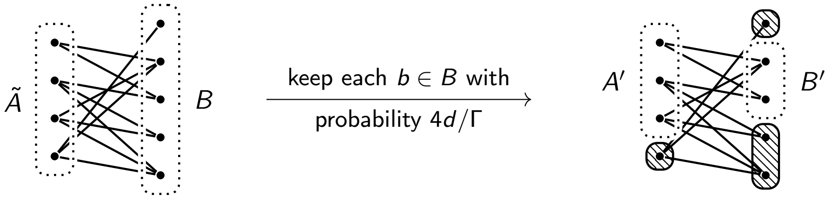


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**Lemma**

*Let $G = (A \cup B, E)$ be a 2-edge-coloured bipartite graph with*

1. $|A| \geqslant 2^{64d+1} \cdot |B|$ *and*
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*Then $G$ contains* **monochromatic induced subgraph** *$H$ with $\delta(H) \geqslant d$.*

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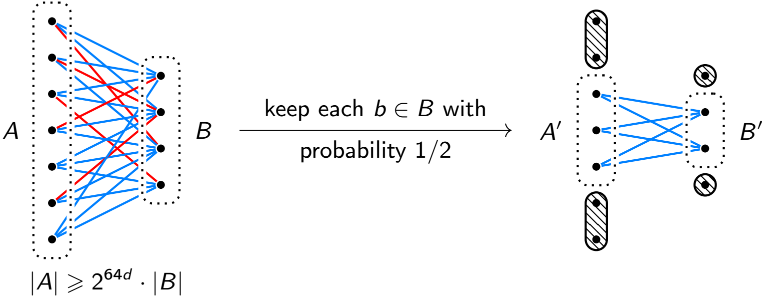


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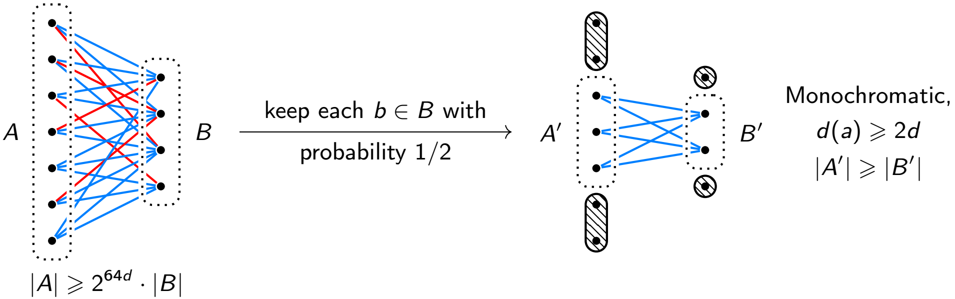


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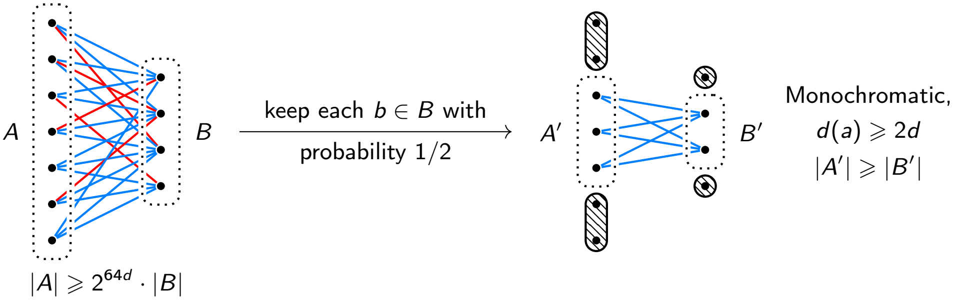


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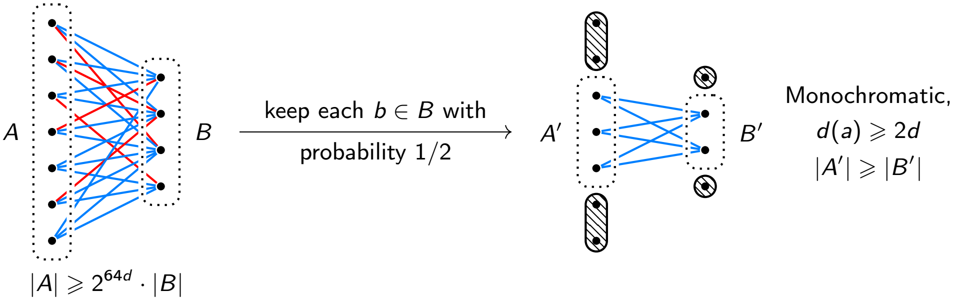


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*Every degree-bounded hereditary class of graphs is polynomially degree-bounded.*

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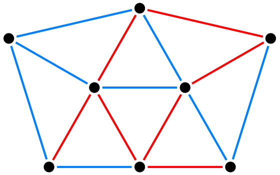

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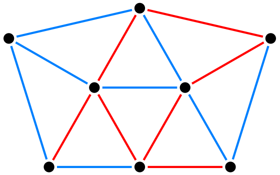

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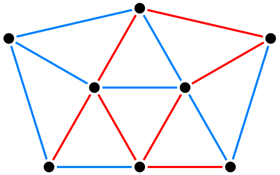

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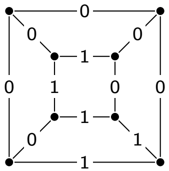

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*The class $\mathcal{EH}$ of even-hole-free graphs is degree-bounded and linearly $\chi$-bounded.*

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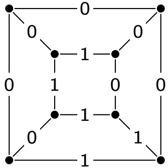

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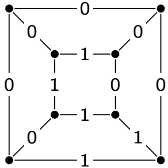


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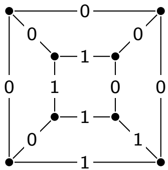


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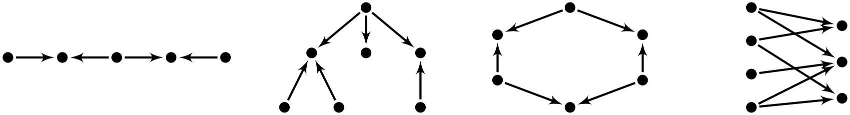

The **Transitive Tournament** on $k$ vertices $TT_k$:




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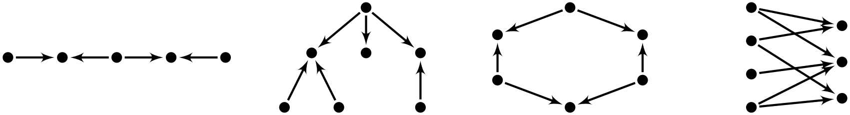

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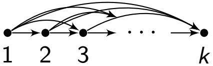


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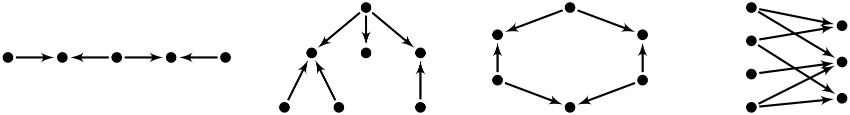

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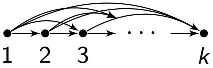


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# Substructures of oriented graphs with large minimum degree

## Corollary

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# Substructures of oriented graphs with large minimum degree

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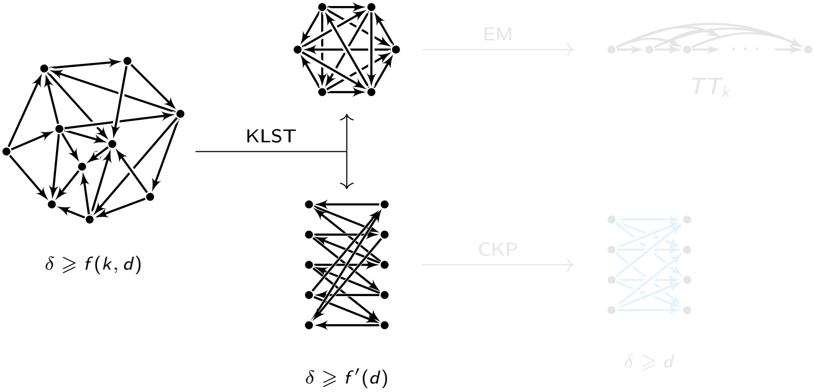

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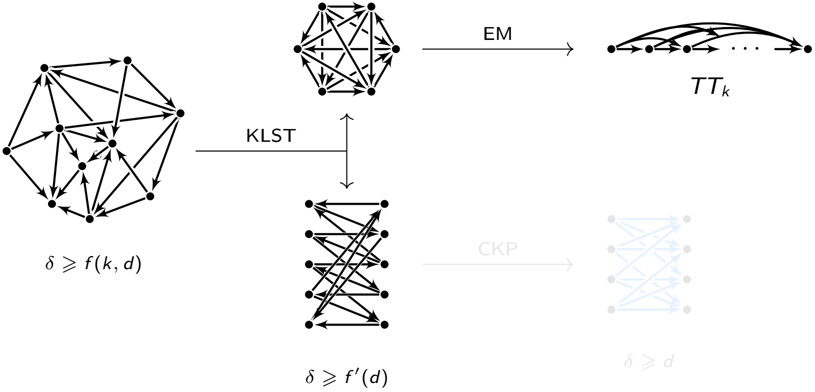

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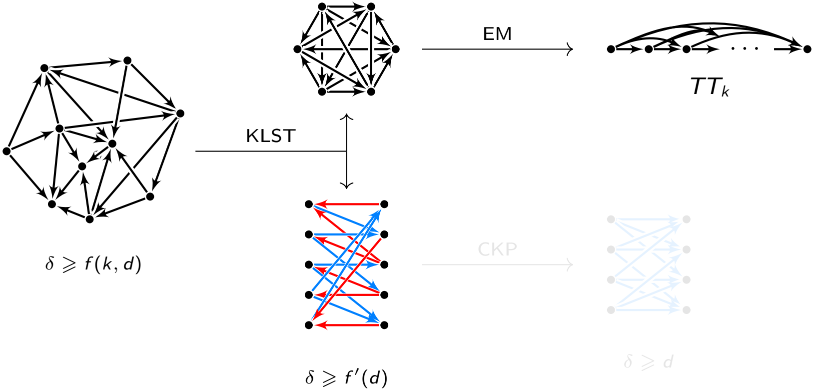

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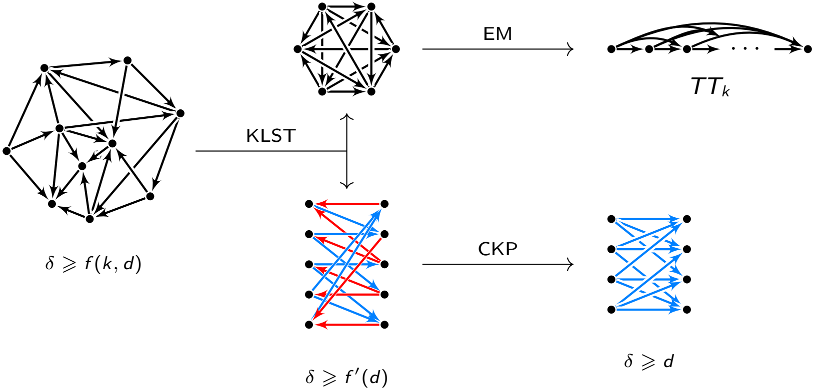

# Application 2: an analogue of Gyárfás-Sumner

**Conjecture (Gyárfás, 1975; Sumner, 1981)**

*The class of $F$-free graphs is $\chi$-bounded if and only if $F$ is a forest.*

**Theorem (Kierstead and Penrice, 1994)**

*The class of $F$-free graphs is degree-bounded if and only if $F$ is a forest.*

For an oriented graph $\vec{F}$, a graph $G$ is $\vec{F}$-free if it has an orientation without any induced copy of $\vec{F}$.

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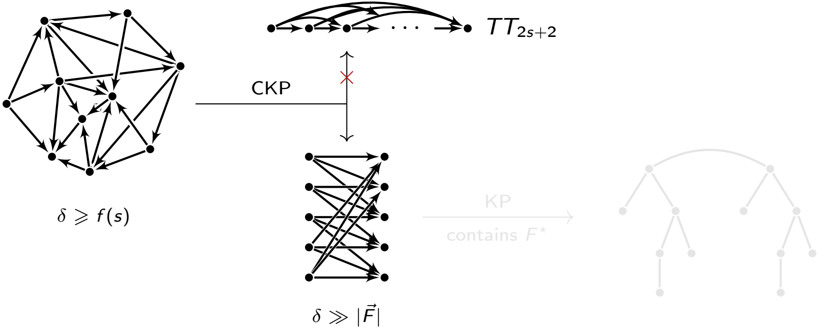

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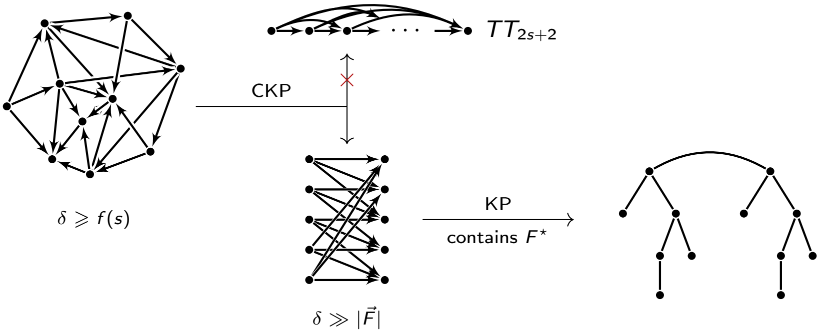

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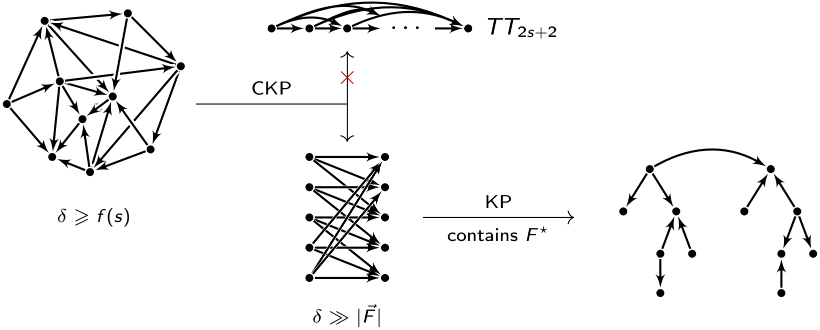

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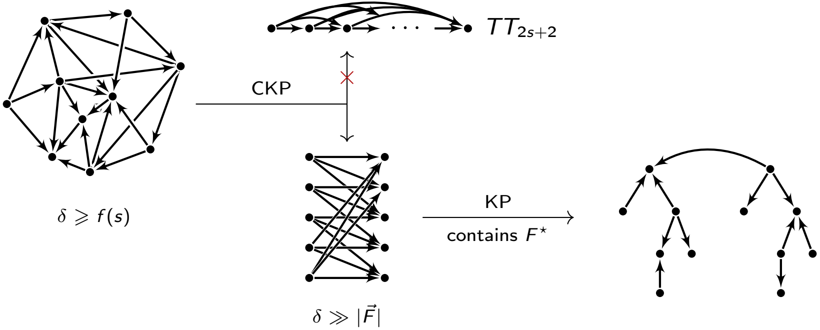

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**Theorem (Kühn and Osthus, 2004)**

*If $\delta(G) \geqslant f(s,t)$ then $G$ contains $K_{s,s}$ or an* **induced even subdivision** *of $K_t$.*

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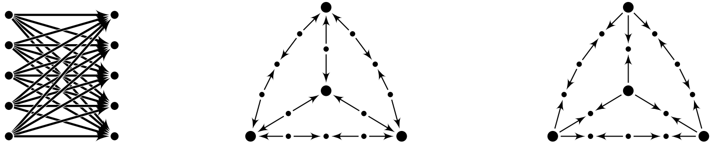

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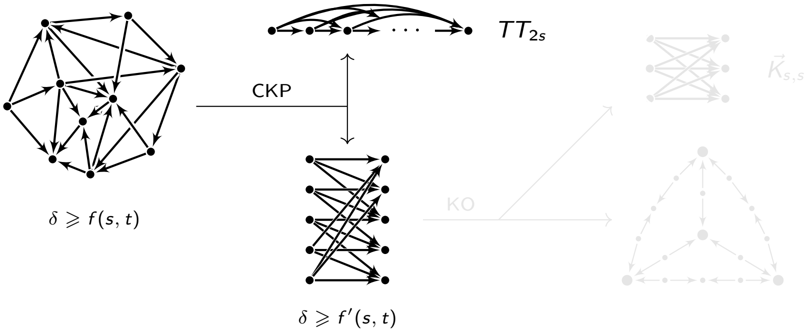

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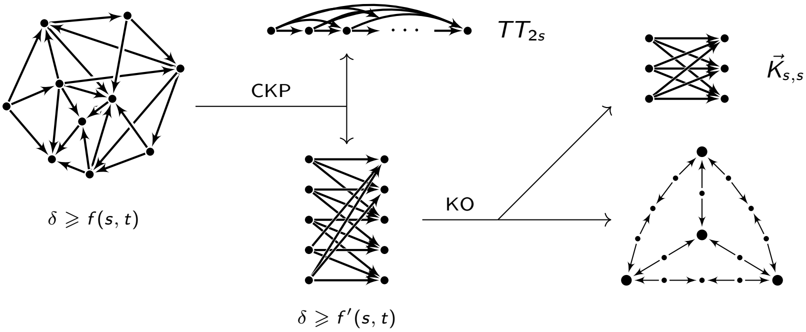

# Application 4: Antidirected cycles and Burling graphs

Let $\mathcal{AC}_{\geqslant \ell}$ be the class of antidirected cycles of length at least $\ell$.

**Corollary**

*For every $\ell$, the class of $\mathcal{AC}_{\geqslant \ell}$-free graphs is degree-bounded.*

*Proof:* Every antidirected subdivision of $K_\ell$ contains an antidirected cycle of length $\geqslant 2\ell$.

**Corollary**

*For every $\ell$, the class of $(\{AC_4\} \cup \mathcal{AC}_{\geqslant \ell})$-free graphs is polynomially $\chi$-bounded.*

*Proof:* It is a hereditary degree-bounded class excluding $K_{3,9}$.

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(Original Proof: [PKKLMTW'14] and [FP'10])

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# Open problem: finding the right function

## Problem

*Find the smallest $f(d)$ such that every 2-edge-coloured graph $G$ with $\delta(G) \geqslant f(d)$ contains a* **monochromatic induced subgraph** *$H$ with $\delta(H) \geqslant d$.*

$$C \cdot 2^{d/2} \leqslant f(d) \leqslant 2^{2^{2^{O(d)}}}$$

# Open problem: other graph parameters

**Theorem (Carbonero, Hompe, Moore, and Spirkl, 2023)**

*There exist 2-edge-coloured graphs $G$ with arbitrarily large chromatic number in which every monochromatic induced subgraph is 4-colourable.*

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*Show that, for every graph $G$ with $\chi(G) \geqslant f(k)$, if $G$ is randomly edge-coloured then*

$$\mathbb{P}\Big(\exists H \subseteq_{\text{ind}} G, \text{ monochromatic, with } \chi(H) \geqslant k\Big) \longrightarrow 1$$

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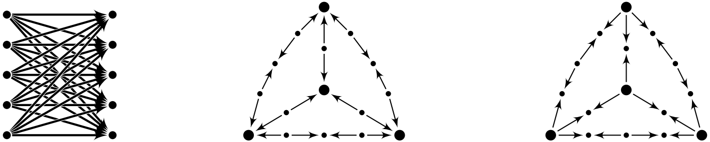

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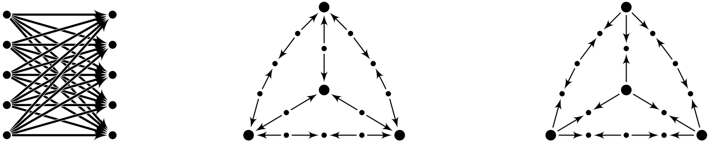

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**Corollary**

*The class of $\vec{F}$-free graphs is degree-bounded if and only if $\vec{F}$ is an antidirected forest.*

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*Characterise the oriented graphs $\vec{F}$ such that $\vec{F}$-free graphs are $\chi$-bounded.*

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