Acyclic Dichromatic Number of Tournaments: these are the Champions

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Abstract

The acyclic dichromatic number of an oriented graph is the minimum size of a vertex-partition such that the digraphs induced by any single part are acyclic, and the oriented bipartite graphs between any two parts are acyclic too. We characterize the subtournaments that must appear in every tournament with sufficiently large acyclic dichromatic number, thereby confirming a conjecture of Bang-Jensen, Picasarri-Arrieta, and Yeo.

1 Introduction

An acyclic colouring of a graph is a proper vertex colouring such that every pair of colour classes induces an acyclic graph, that is, a forest. This notion was introduced in 1973 by Grünbaum [7] who proved that every planar graph has an acyclic 9-colouring, and can thus be decomposed into three induced forests. Borodin then improved this to 5 colours [4], showing that every planar graph can be decomposed into an independent set and two induced forests.

In the setting of directed graphs, vertex colouring has been widely studied since the introduction of the dichromatic number by Erdős [5] and Neumann-Lara [8]. A notable achievement is the characterization of heroes – the subtournaments that must appear in all tournaments with large enough dichromatic number – by Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé [3].

Recently, Bang-Jensen, Picasarri-Arrieta, and Yeo [2] defined a directed analogue to acyclic colouring: an acyclic dicolouring of a digraph is a dicolouring where the arcs between any two colour classes form an acyclic set. They raised the question of which are the tournaments analogous to heroes for this notion, and proposed a conjecture. In this paper, we confirm their conjecture, and characterize the subtournaments that appear in all tournaments with sufficiently large acyclic dichromatic number.

1.1 Definitions and notations

Given a positive integer n, we denote by $[n]$ the set $\{1, \dots, n\}$. All graphs and digraphs in this paper are simple, finite, and contain no parallel or anti-parallel arcs.

Let D be a digraph, $u, v \in V(D)$ two vertices, and $X, Y \subseteq V(D)$ two sets of vertices. If $uv \in E(D)$, we say that u is an *in-neighbour* of v, and that v is an *out-neighbour* of u; we also write $u \rightarrow v$ to denote that uv is an arc of D. Given a vertex $u \in V(D)$, we denote by u^+ the set of its out-neighbours, and by u^- the set its of in-neighbours. We denote by $D[X, Y]$ the subdigraph of D with vertex set $X \cup Y$ and containing every arc of D having one end in X and the other end in Y. In particular, $D[X, X]$ is the subdigraph induced by X, also noted $D[X]$. We write $X \Rightarrow Y$ to say that for all vertices $x \in X$ and

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$y \in Y$ we have $x \to y$. Given two digraphs D_1 and D_2, we denote by $D_1 \Rightarrow D_2$ the digraph obtained from the disjoint union of D_1 and D_2 by adding all arcs from $V(D_1)$ to $V(D_2)$.

A *tournament* is an orientation of a complete graph, that is, an oriented graph with exactly one arc between every pair of vertices. We say that a tournament is *transitive* if it is acyclic, and we denote by TT_n the unique transitive tournament on n vertices. Given three tournaments T_1, T_2, T_3, we denote by $\Delta(T_1, T_2, T_3)$ the tournament obtained from disjoint copies of T_1, T_2, T_3 by adding the remaining arcs so that $T_1 \Rightarrow T_2 \Rightarrow T_3 \Rightarrow T_1$. For the sake of better readability, we may write k in place of TT_k in this last notation; for example, $\Delta(1, k, T)$ stands for $\Delta(TT_1, TT_k, T)$. Given two tournaments H and T, we say that T is *H-free* if it T contains no subtournament isomorphic to H.

A *k-dicolouring* of a digraph D is a function $\varphi \colon V(D) \to [k]$ such that $D[\varphi^{-1}(c)]$ is acyclic for every colour $c \in [k]$. The sets $\varphi^{-1}(c)$ for $c \in [k]$ are called the *colour classes* of φ. A digraph is *k-dicolourable* if it admits a k-dicolouring. The *dichromatic number* of a digraph D, denoted by $\vec{\chi}(D)$, is the least integer k such that D is k-dicolourable. A tournament H is a *hero* if there exists an integer c_H such that every H-free tournament has dichromatic number at most c_H. In a seminal paper, Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé [3] characterized heroes.

Theorem 1 (Berger *et al.* [3])**.** *A tournament H is a hero if and only if:*

- *$H \simeq TT_1$, or*
- *$H \simeq (H_1 \Rightarrow H_2)$, where H_1 and H_2 are heroes, or*
- *$H \simeq \Delta(1, k, H_1)$ or $H \simeq \Delta(1, H_1, k)$, where H_1 is a hero and k a positive integer.*

1.2 Acyclic dicolouring and main result

Let D be a digraph. An *acyclic k-dicolouring* of D is a k-dicolouring with colour classes $V_1, \dots, V_k$ such that the subdigraph $D[V_i, V_j]$ is acyclic for all $i, j \in [k]$. The *acyclic dichromatic number* $\vec{\chi}_{\mathrm{a}}(D)$ of D is the least integer k such that D admits an acyclic k-dicolouring. When $X \subseteq V(D)$ is a set of vertices and D is clear from the context, we may write $\vec{\chi}_{\mathrm{a}}(X)$ in place of $\vec{\chi}_{\mathrm{a}}(D[X])$. A tournament H is a *champion* if there exists an integer c_H such that every H-free tournament has acyclic dichromatic number at most c_H. The acyclic dichromatic number upper-bounds the dichromatic number, so every champion is a hero, but the converse is not true. The following conjecture was posed in [2].

Conjecture 2 (Bang-Jensen, Picasarri-Arrieta, and Yeo [2])**.** *A tournament H is a champion if and only if H is isomorphic to a subtournament of $TT_k \Rightarrow (\Delta(1, 1, k) \Rightarrow TT_k)$ for some integer $k \geqslant 1$.*

In support of their conjecture, Bang-Jensen *et al.* proved that the forward implication holds (*i.e.* every champion has this form), and that the tournaments $\Delta(1, 1, k)$ and $\Delta(1, 1, 1) \Rightarrow TT_1$ are champions. Our main theorem confirms their conjecture.

Theorem 3. *For every integer $k \geqslant 1$, the tournament $TT_k \Rightarrow (\Delta(1, 1, k) \Rightarrow TT_k)$ is a champion.*

Our proof of Theorem 3 relies on the notion of *dimatching.* A *dimatching* is a set of pairwise disjoint arcs $\{a_1b_1, \dots, a_kb_k\}$ such that for each $i, j \in [k]$, we have $a_i \to b_j$ when $i = j$, and $a_i \leftarrow b_j$ when $i \neq j$. In this case, we say that a_i and b_i are *matched* together, and given two vertex sets A and B, we say that this dimatching goes from A to B if $a_i \in A$ and $b_i \in B$ for all $i \in [k]$.

Bang-Jensen *et al.* [2] introduced this structure to obtain 2-dicolourable tournaments with large acyclic dichromatic number. They show that any oriented graph containing a dimatching of size k has acyclic dichromatic number at least $\sqrt{k}$. We conjecture that, conversely, any tournament with sufficiently large acyclic dichromatic number must contain a large dimatching. This is a stronger version of Conjecture 2, as we briefly justify in the next section (see Lemma 8).

Conjecture 4. *There exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for all integers $k \geqslant 1$, every tournament with acyclic dichromatic number at least $f(k)$ contains a dimatching of size k.*

2 Preliminary results

Let us first establish some basic facts about the acyclic dichromatic number. It is easy to see that a vertex colouring of a digraph is an acyclic dicolouring if and only if there are no monochromatic directed cycles, and no directed (even) cycles alternating between two colours. In a tournament, it suffices that there are no monochromatic 3-cycles and no bichromatic alternating 4-cycles.

It is worth mentioning that there exist digraphs with dichromatic number 2 and arbitrarily large acyclic dichromatic number (see [2, Prop. 2]). In particular, given a partition (X, Y) of the vertex set of a digraph D, in general one cannot bound $\vec{\chi}_{\mathrm{a}}(D)$ by any function of $\vec{\chi}_{\mathrm{a}}(X)$ and $\vec{\chi}_{\mathrm{a}}(Y)$. Along the proof, we overcome this difficulty by repeatedly applying the following lemma.

Lemma 5. *Let D be a digraph and H be an induced subdigraph of D, then*

$$\vec{\chi}_{\mathrm{a}}(D) \leqslant \vec{\chi}_{\mathrm{a}}\big(D \setminus E(H)\big) \cdot \big(\vec{\chi}_{\mathrm{a}}(H) + 1\big).$$

Proof. Let φ_D and φ_H be any acyclic dicolourings of $D \setminus E(H)$ and H, respectively. We show that the colouring $\varphi^\star$ of D defined below is an acyclic dicolouring of D, hence implying the result.

$$\varphi^\star \colon v \mapsto \begin{cases} \big(\varphi_D(v), \varphi_H(v)\big) & \text{if } v \in V(H) \\ \big(\varphi_D(v), 0\big) & \text{otherwise.} \end{cases}$$

Suppose for contradiction that D, coloured with $\varphi^\star$, contains a directed cycle C which is monochromatic or bichromatic with alternating between two colours. Then the same holds with respect to φ_D. By the choice of φ_D, the cycle C uses at least one arc uv of H. Since C is monochromatic or alternating between two colours, every vertex w of C has the same colour as u or as v. Hence, every vertex w of C satisfies $\varphi_H(w) \neq 0$, and since H is an induced subdigraph of D, the cycle C is in H. But then C is monochromatic or alternating between two colours with regard to φ_H, a contradiction. ☐

By iterating the lemma above, we obtain the following useful corollary. Informally, it states that if the edges of a digraph D can be covered (up to an acyclic set) by a bounded number of induced subdigraphs with bounded acyclic dichromatic number, then the acyclic dichromatic number of D is also bounded.

Corollary 6. *Let D be a digraph and $D_1, \dots, D_\ell$ be a collection of induced subdigraphs of D such that $D \setminus \bigcup_{i=1}^{\ell} A(D_i)$ is acyclic, then*

$$\vec{\chi}_{\mathrm{a}}(D) \leqslant \prod_{i=1}^{\ell} \Big(\vec{\chi}_{\mathrm{a}}(D_i) + 1\Big).$$

We further show that we can find any champion in every tournament that contains a sufficiently large dimatching. For this, we make use of the following analogue of Ramsey's theorem for tournaments.

Theorem 7 (Erdős and Moser [6])**.** *For all integers $n \geqslant 0$, every tournament of order 2^n contains a transitive tournament of order $n+1$.*

Lemma 8. *Let T be a tournament and $k \geqslant 1$ be an integer. If T contains a dimatching of size $2^{2k} + 2^k + 4k$, then T contains a subtournament isomorphic to $TT_k \Rightarrow (\Delta(1,1,k) \Rightarrow TT_k)$.*

Proof. Let M be a dimatching of size $2^{2k} + 2^k + 4k$ in T. Denote A and B the vertex sets of size $2^{2k} + 2^k + 4k$ such that M goes from A to B. Let $B' \subset B$ denote the vertices of B with at least $2^{k-1} + 2k$ in-neighbours in B, and let A' be the vertices of A matched with vertices in B'. It is well known that, for every integer $\ell \geqslant 1$, every tournament contains at most $2\ell - 1$ vertices of in-degree less than ℓ (see [1, Prop. 2.2.2]). In particular, for $\ell = 2^{k-1} + 2k$, we obtain that $|A'| = |B'| \geqslant 2^{2k}$.

By Theorem 7, there exists a subset $A'' \subset A'$ of size $2k+1$ that induces a transitive tournament in T. Let $a \in A''$ be the "middle" vertex in this transitive tournament (that is, the unique vertex with

k in-neighbours and k out-neighbours in A'') and let b be the vertex matched to a. By definition of B', the vertex b has at least $2^{k-1}+2k$ in-neighbours in B. At least 2^{k-1} of them are not matched to any vertex in A'', so by Theorem 7, there exists a set $B'' \subseteq B$ consisting of k in-neighbours of b not matched to any vertex in A'', and that induces a transitive tournament in T. It is easy to see that $A'' \cup B'' \cup \{b\}$ induces a tournament isomorphic to $TT_k \Rightarrow (\Delta(1,1,k) \Rightarrow TT_k)$. □

A graph $G=(V,E)$ is an *interval graph* if there exists a family $(I_v)_{v\in V}$ of intervals of the real line such that, for every $u,v \in V$, we have $uv \in E$ if and only if $I_u \cap I_v \neq \varnothing$. We make use of the following technical lemma about interval graphs. It captures the fact that, in a connected interval graph, starting a Breadth-First Search from the interval with smallest right endpoint yields layers that are naturally ordered along the real line.

Lemma 9. *Let G be an interval graph, where for each vertex $v \in V(G)$, the interval associated to v is denoted by I_v. There exists a partition $L_0,\dots,L_h$ of V such that:*

- *for each $0 \leqslant i \leqslant h$, either $|L_i|=1$, or $i \geqslant 1$ and there exists $u \in L_{i-1}$ such that $L_i \subseteq N_G(u)$; and*
- *for all $0 \leqslant i \leqslant j-2 \leqslant h-2$, we have $\max\Big(\bigcup_{v\in L_i} I_v\Big) < \min\Big(\bigcup_{v\in L_j} I_v\Big)$.*

Proof. Let $\prec$ denote a total order on $V(G)$ defined, for $u,v \in V(G)$, by $u \prec v$ if $\max(I_u) < \max(I_v)$, and in case of equality between these maxima, we order u and v arbitrarily. It is classical and easy to observe that this order satisfies the following property for all $u,v,w \in V(G)$:

$$\text{if } u \prec v \prec w \text{ and } uw \text{ is an edge of } G\text{, then } vw \text{ is an edge of } G. \tag{1}$$

Let a_0 be the smallest vertex in G (with respect to $\prec$), and as long as a_i has neighbours larger than itself (with respect to $\prec$), define a_{i+1} as its largest neighbour. Let a_d be the maximum such a_i. By (1), observe that every vertex between a_{i-1} and a_i is a neighbour of a_i, so all vertices smaller than a_d belong to the connected component C of a_0. Moreover, all vertices after a_d belong to another connected component, as if a vertex after a_d is adjacent to C, then it is adjacent to a_d by (1), but this contradicts the maximality of a_d. Observe that a_i is at distance at most i from a_0 (we prove below that the distance is exactly i), thus every vertex in C is at distance at most $d+1$ from a_0. For each $0 \leqslant i \leqslant d+1$, define L_i as the set of vertices at distance i from a_0. Note that $L_0=\{a_0\}$.

Claim 9.1. *For every $1 \leqslant i \leqslant d$, $a_i \in L_i$, $L_i \subseteq N_G(a_{i-1})$, $L_i \preceq a_i$ and if $i \geqslant 2$ then $a_{i-2} \prec L_i$.*

Proof of claim. We proceed by induction on i. It is trivial for $i=1$ by definition of a_1 and L_1. Assume now that $i \geqslant 2$ and that it is true for $j<i$. Assume first, for a contradiction, that $a_i \in L_j$ for $j<i$. By induction hypothesis $L_j \subseteq N_G(a_{j-1})$, which contradicts the choice of a_j since $a_j \prec a_i$. Since a_i is at distance at most i from a_0, we get $a_i \in L_i$.

Now let a be an arbitrary vertex in L_i. Let $j \geqslant 1$ be such that $a_{j-1} \prec a \preceq a_j$. Note that such a j exists as every vertex $u \in C$ satisfies $a_0 \preceq u \preceq a_d$. By (1), a is a neighbour of a_j.

If $j \leqslant i-2$, by induction $a_j \in L_j$ and a is at distance at most $j+1<i$ from a_0, a contradiction. This shows that $a_{i-2} \prec L_i$. If $j=i-1$ then a is a neighbour of a_{i-1}, as desired. If $j \geqslant i$, let $a' \in L_{i-1}$ be such that $a'a$ is an edge. By induction $a_{i-2}a'$ is an edge so by definition of a_{i-1} we have $a' \preceq a_{i-1} \prec a$. Now by (1) we have $a \in N_G(a_{i-1})$ as desired. This shows $L_i \subseteq N_G(a_{i-1})$. Now the definition of a_i implies that $L_i \prec a_i$. ♢

As observed above, the connected component of a_0 is precisely the set of vertices between a_0 and a_d, so if there are still remaining vertices, we can start again with the successor of a_d, and perform the same BFS to get the successive layers, and the sequence of partitions satisfies the requirement of the Lemma. □

3 Proof of Theorem 3

This section is dedicated to the proof of Theorem 3. We first reduce to the case of bipartite tournaments (*i.e.* orientations of complete bipartite graphs), then we make use of the following notion.

Let $T = (A, B, E)$ be a bipartite tournament. Given an ordering $\prec$ of B, we say that a vertex $a \in A$ *switches* at a vertex $b \in B$ if $b' \to a \to b$ or $b' \leftarrow a \leftarrow b$, where b' is the predecessor of b in $\prec$. Informally, it means that the orientations of the arcs at a change, along the ordering of B, when arriving at b.

The idea of the proof is that, in a bipartite tournament T with large acyclic dichromatic number, either we can find a vertex in A that switches many times with respect to some ordering of B, or T contains a large dimatching. In both cases, we can conclude that T contains $TT_k \Rightarrow (\Delta(1,1,k) \Rightarrow TT_k)$. The proof of Theorem 3 thus relies on the following key lemma.

Lemma 10. *There exists a function $f\colon \mathbb{N}^3 \to \mathbb{N}$ such that, for all integers $p, q, r \geqslant 1$ and every bipartite tournament $T = (A, B, E)$ with $\vec{\chi}_{\mathrm{a}}(T) \geqslant f(p, q, r)$, at least one of the following holds:*

(i) for every ordering of B, some vertex of A switches at least r times; or

(ii) T contains a dimatching of size p from A to B; or

(iii) T contains a dimatching of size q from B to A.

Proof. Let $p, q, r \geqslant 1$ be positive integers and $T = (A, B, E)$ a bipartite tournament such that none of (i), (ii), and (iii) hold. We show that the acyclic dichromatic number of T is bounded by some value $f(p, q, r)$ to be determined, which implies the desired result. We proceed by induction on $p + q + r$.

If $\min\{p, q, r\} = 1$, then we claim that T has no directed cycle. Indeed, a vertex of A on a directed cycle has at least one in-neighbour and one out-neighbour in B, and such a vertex switches at least once with regard to every ordering of B. Moreover, any arc from A to B and any arc from B to A forms a dimatching of size 1. Thus, if $\min\{p, q, r\} = 1$, then T is acyclic, and we can set $f(p, q, r) = 2$.

Now suppose $p, q, r > 1$. Since (i) does not hold, there exists an ordering $\prec$ on B such that every vertex of A switches strictly less than r times with respect to $\prec$. In the following, when speaking about vertices in B, we use the words smaller and larger to denote their relation with respect to $\prec$, and use the notations $[b, b']$ (or their variants with open brackets such as $[b, b'[$) to denote intervals in B with regard to $\prec$. We denote respectively $b_{\min}$ and $b_{\max}$ the smallest and largest element of B. Let us partition A into four sets depending on the orientation of the arcs at $b_{\min}$ and $b_{\max}$:

- $A^{++} = b^+_{\min} \cap b^+_{\max}$,
- $A^{+-} = b^+_{\min} \cap b^-_{\max}$,
- $A^{-+} = b^-_{\min} \cap b^+_{\max}$, and
- $A^{--} = b^-_{\min} \cap b^-_{\max}$.

In what follows, we bound the acyclic dichromatic number of the bipartite tournament induced by each of these sets together with B. The result then follows from Corollary 6. Remark that if a vertex $a \in A$ never switches then it lies in no directed cycle, so removing it does not decrease the acyclic dichromatic number, thus without loss of generality, we may assume that every vertex $a \in A$ switches at least one time. The key object in our proof is the interval graph that we now define. For each vertex $a \in A$, denote respectively $\mathrm{f}(a)$ and $\ell(a)$ the first and last switches of a in B, and let I_a be the interval

$$I_a \coloneqq [\mathrm{f}(a), \ell(a)[= \{b \in B : \mathrm{f}(a) \preceq b \prec \ell(a)\}.$$

Let G be the interval graph on A defined by these intervals (*i.e.* G has vertex set A, and for $a, a' \in A$, there is an edge aa' in G if and only if I_a and $I_{a'}$ intersect).

Claim 10.1. *If $K \subseteq A$ is a clique of G, then $\vec{\chi}_{\mathrm{a}}\big(T[K \cup B]\big) \leqslant f(p, q, r-1)^2$.*

Proof of claim. Let $a \in K$ be such that $f(a)$ is maximal. Then for $a' \in K$, we have $\mathrm{f}(a') \preceq f(a) \prec \ell(a)$, where the first inequality holds by definition of a, and the second because otherwise a and a' are not adjacent in G. Hence each $a' \in K$ switches at least once in $[b_{\min}, f(a)]$ and once in $]f(a); b_{\max}]$, so a' switches less than $r-1$ times in $[b_{\min}, f(a)]$ and in $]f(a); b_{\max}]$. By induction hypothesis, this implies

$$\vec{\chi}_{\mathrm{a}}\big(K \cup [b_{\min}, f(a)]\big) \leqslant f(p,q,r-1) - 1 \quad \text{and} \quad \vec{\chi}_{\mathrm{a}}\big(K \cup]f(a), b_{\max}]\big) \leqslant f(p,q,r-1) - 1,$$

and the claim follows by applying Corollary 6. ◊

Claim 10.2. *A stable set in A^{++} (resp. A^{--}) has size less than p (resp. q).*

Proof of claim. Suppose S is a stable set in A^{++}. Let $s, s' \in S$ be distinct vertices. Recall that $b_{\min} \to s$ and $b_{\max} \to s$, because $s \in A^{++}$, and as $\mathrm{f}(s)$ is the first switch of s, it holds that $s \to \mathrm{f}(s)$. By definition of $I_{s'}$, we have $\mathrm{f}(s') \in I_{s'}$, thus $\mathrm{f}(s') \notin I_s$ as otherwise s and s' are adjacent in G. Hence either $\mathrm{f}(s') \prec \mathrm{f}(s)$ or $\ell(s) \preceq \mathrm{f}(s')$. If $\mathrm{f}(s') \prec \mathrm{f}(s)$, then note that $b_{\min} \to s$ and s does not switch in $[b_{\min}, \mathrm{f}(s)[$, so $\mathrm{f}(s') \to s$. If $\ell(s) \preceq \mathrm{f}(s')$, then note that $b_{\max} \to s$ and s does not switch in $[\ell(s), b_{\max}]$, so $\mathrm{f}(s') \to s$. It follows that the arcs $\{s\ \mathrm{f}(s)\}_{s \in S}$ form a dimatching from A to B, so $|S| < p$ by hypothesis.

The proof for A^{--} works symmetrically. ◊

Claim 10.3. $\vec{\chi}_{\mathrm{a}}\big(A^{++} \cup A^{--} \cup B\big) \leqslant \big(f(p,q,r-1)^2 + 1\big)^{p+q}$

Proof of claim. Since any interval graph is perfect, its vertices can be covered by a number of cliques equal to its stability number. Hence by Claim 10.2, we can partition A^{++} and A^{--} into p and q cliques, respectively. Then the claim follows by combining Claim 10.1 with Corollary 6. ◊

It remains to bound $\vec{\chi}_{\mathrm{a}}(A^{+-} \cup B)$ and $\vec{\chi}_{\mathrm{a}}(A^{-+} \cup B)$. This time, G may have unbounded stability number, so we use a different strategy based on the partition guaranteed by Lemma 9.

Claim 10.4. *Let b_1 and b_2 be vertices in B such that $(b_1^- \cap b_2^+) \neq \varnothing$. Then*

$$\vec{\chi}_{\mathrm{a}}\Big((b_1^+ \cap b_2^-) \cup B\Big) \leqslant f(p-1,q,r) \cdot f(p,q-1,r).$$

Proof of claim. Let $a \in (b_1^- \cap b_2^+)$. Observe that every dimatching from $(b_1^+ \cap b_2^-)$ to a^- can be augmented by the arc ab_1, so by induction hypothesis, it holds that

$$\vec{\chi}_{\mathrm{a}}\big((b_1^+ \cap b_2^-) \cup a^-\big) \leqslant f(p-1,q,r) - 1.$$

Similarly, every dimatching from a^+ to $(b_1^+ \cap b_2^-)$ can be augmented by the arc b_2a, and thus

$$\vec{\chi}_{\mathrm{a}}\big((b_1^+ \cap b_2^-) \cup a^+\big) \leqslant f(p,q-1,r) - 1.$$

The claim follows from these two inequalities together with Corollary 6. ◊

The previous claim applied to $b_{\min}$ and $b_{\max}$ shows that if both A^{+-} and A^{-+} are both nonempty, then we are done. Hence (up to reversing all the arcs, and swapping p and q) we can assume that A^{-+} is empty. It remains to bound $\vec{\chi}_{\mathrm{a}}(A^{+-} \cup B)$. For better readability, we denote $G[A^{+-}]$ by G^{+-}.

Let $a \in A^{+-}$. By definition of $\mathrm{f}(a)$ and using that $b_{\min} \to a$, we have $b \to a$ for $b \prec \mathrm{f}(a)$, and $a \to \mathrm{f}(a)$. Likewise, by definition of $\ell(a)$ and using that $a \to b_{\max}$, we have $a \to b$ for $b \succeq \ell(a)$, and $b \to a$ when b is the predecessor of $\ell(a)$ in $\prec$.

Claim 10.5. *For every $a \in A^{+-}$,*

$$\vec{\chi}_{\mathrm{a}}\big(N_{G^{+-}}(a) \cup B\big) \leqslant \big(f(p,q,r-1)^2 + 1\big)^2 \cdot \big(f(p-1,q,r) \cdot f(p,q-1,r) + 1\big).$$

Proof of claim. Let $a \in A$, and denote $b \in B$ the vertex preceding $\ell(a)$ in $\prec$. We define:

- $A_1 = \{a' \in A^{+-} : \mathrm{f}(a') \preceq \mathrm{f}(a) \prec \ell(a')\}$,

- $A_2 = \{a' \in A^{+-} : \mathrm{f}(a') \prec \ell(a) \preceq \ell(a')\}$, and
- $A_3 = \{a' \in A^{+-} : \mathrm{f}(a) \prec \mathrm{f}(a') \prec \ell(a') \prec \ell(a)\}$.

Note that A_1 and A_2 are cliques in G, since all intervals $I_{a'}$ for $a' \in A_1$ (resp. $a' \in A_2$) contain $\mathrm{f}(a)$ (resp. b). Therefore, using Claim 10.1 and Corollary 6, we get

$$\vec{\chi}_{\mathrm{a}}\big(A_1 \cup B\big) \leqslant f(p,q,r-1)^2 \qquad \text{and} \qquad \vec{\chi}_{\mathrm{a}}\big(A_2 \cup B\big) \leqslant f(p,q,r-1)^2.$$

For $a' \in A_3$, we have $\mathrm{f}(a) \to a'$ as $\mathrm{f}(a) \prec \mathrm{f}(a')$, and $a' \to b$ as $\ell(a') \preceq b$, thus $A_3 \subseteq (f(a)^+ \cap b^-)$. Recall that $a \to \mathrm{f}(a)$ and $b \to a$, so $(f(a)^- \cap b^+) \neq \varnothing$. Hence, we can apply Claim 10.4 to $\mathrm{f}(a)$ and b, which yields

$$\vec{\chi}_{\mathrm{a}}\big(A_3 \cup B\big) \leqslant f(p-1,q,r) \cdot f(p,q-1,r).$$

The claim follows from Corollary 6 and the fact that $A_1 \cup A_2 \cup A_3 = N_{G^{+-}}(a)$. ♢

Let $L_0, \dots, L_h$ be the partition of $V(G^{+-})$ guaranteed by Lemma 9. For each $0 \leqslant i \leqslant h$, denote $B_i = \bigcup_{a \in L_i} I_a$. By Lemma 9, for $0 \leqslant i \leqslant j-2 \leqslant h-2$, we have $\max(B_i) \prec \min(B_j)$ so B_i and B_j are disjoint. To make use of that, we split the layers in two sets according to their parity:

$$L_{\text{even}} = \bigcup_{i \text{ even}} L_i \qquad \text{and} \qquad L_{\text{odd}} = \bigcup_{i \text{ odd}} L_i.$$

Let $a \in L_i$ and $b \in B$. If $b \prec \min(B_i)$ then $b \prec \mathrm{f}(a)$ so $b \to a$, and if $\max(B_i) \prec b$ then $\ell(a) \preceq b$ so $a \to b$. Is is easy to prove that that every directed cycle of $T[L_{\text{even}} \cup B]$ is included in $T[L_i \cup B_i]$ for some i (even). Hence, putting together acyclic dicolourings of $T[L_i \cup B_i]$ for each even index i, yields an acyclic dicolouring of $T[L_{\text{even}} \cup B]$, thus

$$\vec{\chi}_{\mathrm{a}}(L_{\text{even}} \cup B) \leqslant \max_{i \text{ even}} \vec{\chi}_{\mathrm{a}}(L_i \cup B_i).$$

The same holds for L_{odd} and odd layers, so by Corollary 6 we get

$$\vec{\chi}_{\mathrm{a}}(A^{+-} \cup B) \leqslant \Big(\max_{0 \leqslant i \leqslant h} \vec{\chi}_{\mathrm{a}}(L_i \cup B_i) + 1\Big)^2.$$

It thus suffices to bound $\vec{\chi}_{\mathrm{a}}(L_i \cup B_i)$ for $0 \leqslant i \leqslant h$. By Lemma 9, for each $0 \leqslant i \leqslant h$, either $|L_i| = 1$, or $i \geqslant 1$ and $L_i \subseteq N_{G^{+-}}(u)$ for some $u \in L_{i-1}$. Hence by Claim 10.5, we have

$$\vec{\chi}_{\mathrm{a}}(L_i \cup B_i) \leqslant \big(f(p,q,r-1)^2 + 1\big)^2 \cdot \big(f(p-1,q,r) \cdot f(p,q-1,r) + 1\big).$$

This concludes the proof of the lemma. □

We now derive our main theorem, which we restate here for convenience.

Theorem 3. *For every integer $k \geqslant 1$, the tournament $TT_k \Rightarrow (\Delta(1,1,k) \Rightarrow TT_k)$ is a champion.*

Proof. Denote by C_k the tournament $TT_k \Rightarrow (\Delta(1,1,k) \Rightarrow TT_k)$, and let T be a C_k-free tournament. The goal is to show that the acyclic dichromatic number of T is bounded by some value $g(k)$. By Theorem 1, T_k is a hero, so there exists an integer $h(k)$ such that $\vec{\chi}(T) \leqslant h(k)$. Hence, we can partition T into $c \leqslant h(k)$ transitive tournaments $T_1, \dots, T_c$. Remark that the subtournaments induced by two colour classes $\big\{T[T_i \cup T_j]\big\}_{i,j \in [c]}$ together cover all arcs of T. Thus, by Corollary 6,

$$\vec{\chi}_{\mathrm{a}}(T) \leqslant \prod_{i,j \in [c]} \big(\vec{\chi}_{\mathrm{a}}(T_i \cup T_j) + 1\big).$$

Therefore, it suffices to bound the acyclic dichromatic number of $T[T_i \cup T_j]$ for each $i,j \in [c]$. Fix $i,j \in [c]$. Note that $T[T_i]$ and $T[T_j]$ are acyclic induced subtournaments of $T[T_i \cup T_j]$, and after removing their edges, we are left with the bipartite tournament $T[T_i, T_j]$, so by Corollary 6, we have $\vec{\chi}_{\rm a}(T_i \cup T_j) \leqslant 4 \cdot \vec{\chi}_{\rm a}\big(T[T_i, T_j]\big)$. Denote $A = T_i$, $B = T_j$, and $T' = T[T_i, T_j]$. Let f be a function that satisfies the statement of Lemma 10, and towards a contradiction, suppose that

$$\vec{\chi}_{\rm a}(T') \geqslant f\big(r(k), r(k), 6k\big),$$

where $r(k) = 2^{2k} + 2^k + 4k$. According to Lemma 10, either T' contains a dimatching of size $r(k)$ or for every ordering of B there is a vertex $a \in A$ that switches at least at $6k$ times. If T' contains a dimatching of size $r(k)$, then T contains C_k by Lemma 8.

Hence, assume that for every ordering of B, there is a vertex $a \in A$ that switches at least at $6k$ distinct indices. Denote $\prec$ the topological ordering of B, so that we have $b \prec b'$ if and only if $b \to b'$. Let $a \in A$ be a vertex that switches at $6k$ different vertices $b_1 \prec \cdots \prec b_{6k}$. For each $j \in [6k]$, we denote respectively b_j^- and b_j^+ the in-neighbour and the out-neighbour of a among b_j and its predecessor in $\prec$. Remark that b_j^- and b_{j+1}^- may not be distinct, but b_j^- and b_{j+2}^- are always distinct. Let

$$X = \{a\} \cup \{b_2^-, b_4^-, \dots, b_{2k}^-\} \cup \{b_{2k+1}^+\} \cup \{b_{2k+2}^-, b_{2k+4}^-, \dots, b_{4k}^-\} \cup \{b_{4k+2}^+, b_{4k+4}^+, \dots, b_{6k}^+\}.$$

Observe that the relations below hold, and thus X induces a copy of C_k, a contradiction.

$$\{b_2^-, b_4^-, \dots, b_{2k}^-\} \Rightarrow \Big(X \setminus \{b_2^-, b_4^-, \dots, b_{2k}^-\}\Big),$$

$$a \Rightarrow b_{2k+1}^+ \Rightarrow \{b_{2k+2}^-, b_{2k+4}^-, \dots, b_{4k}^-\} \Rightarrow a, \text{ and}$$

$$\Big(X \setminus \{b_{4k+2}^+, b_{4k+4}^+, \dots, b_{6k}^+\}\Big) \Rightarrow \{b_{4k+2}^+, b_{4k+4}^+, \dots, b_{6k}^+\}.$$

Hence $\vec{\chi}_{\rm a}(T') < f\big(r(k), r(k), 6k\big)$. It follows that the acyclic dichromatic number of T is at most

$$g(k) := \big(4 \cdot f\big(r(k), r(k), 6k\big) + 1\big)^{h(k)^2}.$$

This concludes the proof of the theorem. □

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